Direct Algorithm for the Calculation of Vertical Displacements and Deformations of Constructions Using High-Precision Geometric Leveling

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Abstract: *The analysis of vertical displacements and deformations (in the form of either settlements or uplifts) is essential for the study of constructions behavior. In practice, these values are obtained using measurements of control points (settling benchmarks) positioned on buildings. A series of cyclic measurements are conducted in order to determine the elevation of control points with respect to several fixed point of the local reference networkThe paper is first introducing a proposed methodology for establishing the accuracy of measuring procedures of leveling measurements in the field, prior to performing them. This is required in order to ensure the proper precision which is specific to each construction. The paper is further presenting an improved algorithm for calculating the vertical displacements and deformations as a direct function of the variations of level differences measured during each cycle, with respect to the initial (precedent) and the current cycle. Finally, the paper is introducing a mathematical model for the evaluation of the accuracy of the results.*

1. Introduction

When analyzing the in-situ behavior of construction, the data which describe the vertical deformations and displacements are essential. Following a sign convention, settlements are assigned negative values while uplifts have positive values.

In order to determine the vertical displacements and deformations, one should cyclically determine the elevation of the control points (mobile marks or, as they are sometimes named, settlement marks) which are fixed on the construction. Analyzing the time history of the displacements of these control points, one can identify the changes in the veritical position of the construction. When studying constructions behavior during both the construction phase as well its service life, geodetic methods are extensively used as they ensure a high accuracy of measurements performed for the determination of displacements and deformations.

The estimation of vertical displacements and deformations of aq construction and of the surrounding foundation soil is performed according to a special project which specifies the location of the control points on the building, as well as the location of the fixed points which represent the local leveling network which is used as a reference system for the determination of the absolute vertical displacements and deformations.

The geometric leveling is achieved in compliance with the requirements of the technical standard imposed by the national geometric leveling of the order 0, I or II. The geometric leveling system is formed of polygons which incorporate the fixed points (benchmarks) of the reference network, the control points and the intermediate points. At the same time, the station mid-points for geometric leveling are set out in such a way that their position remains unchanged for each measurement cycle. The selection of the topographic instruments and of the geometrical leveling methods is made in order to ensure the accuracy required by each specific construction. Depending on the size and importance of the construction, a compensation of the leveling and/or networks is performed using rigorous or semi-rigorous methods combined with the method of the least squares.

The paper is presenting an improved algorithm for the calculation of vertical deformations and displacements of constructions as a function of the level differences measured during each observation cycle. The field leveling measurements are conducted on the basis of a preliminary study, according to the imposed requirements in terms of accuracy.

2. Preliminary Study on the Required Accuracy of Field Measurements

The fundamental problem in determining the vertical displacements and deformations of a construction is related to the estimation of the actual (real) accuracy of the measured elements. The accuracy level has to comply with the requirements imposed by each particular case. The present section describes a procedure which enables the evaluation of the required accuracy of the geometric leveling used to measure the vertical displacements and deformations.

The geometrical leveling measurements are performed using high accuracy levels equipped with optical micrometers and with the aid of high precision (invar) measuring rods. Of the several methods used to determine the level differences in the field, the authors chose the most widely accepted one, in which level differences are measured with one level and readings performed on both scales of the invar rod use two different horizons of the instrument. This procedure is strongly related to the need to ensure the proper accuracy required to measure the level differences in the field. In the case of the mid-point geometric leveling between two fixed points, R_1 and R_2, the vertical displacement of the control point, j, $\left(j=\overline{1,n-1}\right)$, in the t cycle of measurement $\left(t=\overline{1,n}\right)$, is expressed as

$$\pm \Delta H_j^t = H_j^t - H_j^0 \tag{1}$$

where: H_j^0 is the elevation of the point during the initial cycle (the “zero” cycle); H_j^t is the elevation of the same point measured during the t-cycle.

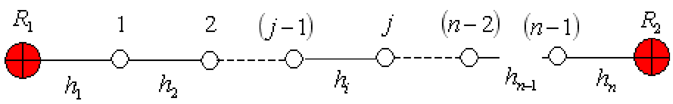

Fig. 1 Schema of the geometrical leveling network

The mean square error of the vertical displacement for the considered point is

$$s_{\Delta H_j^t} = \sqrt{s^2_{H_j^t} + s^2_{H_j^0}} \tag{2}$$

Considering that the magnitudes of the elevation points have been determined with almost equal precision, $s_{H_j^0} \cong s_{H_j^t} \cong s_{H_j}$, Eq. (2) becomes,

$$s_{\Delta H_j^t} \approx \sqrt{s^2_{H_j} + s^2_{H_j}} = s_{H_j}\sqrt{2} \tag{3}$$

For the case of the mid-point geometrical leveling with fixed support points (benchmark points) of known elevation, the means square error of the control point, j, is expressed as

$$s_{H_j} = \frac{s_h}{\sqrt{p_j}} \tag{4}$$

where: s_H is the mean square error of the level difference determined in the station point; p_j is the weight of the considered control point which is expressed as,

$$p_j = \frac{n}{n_i(n-n_i)} \quad , \quad i = \overline{1,n} \tag{5}$$

In the above expression, n is the total number of station points; n_i is the number of station points located between the control point, j, and the initial (support) point. Replacing Eq. (4) into Eq. (3), the mean square error of the vertical displacement is obtained as,

$$s_{\Delta H_j} = \frac{s_h\sqrt{2}}{\sqrt{p_j}} = s_h\sqrt{\frac{2n_i(n-n_i)}{n}} \tag{6}$$

For each construction under evaluation, one should therefore anticipate the required level of accuracy of the vertical displacement and deformation for the control point which is most affected by displacement. An important question is: what is the required accuracy for the measured level differences between the control points, so that the vertical displacement of the most affected point is determined with an accuracy which is higher then the initially imposed one? The authors are further addressing this issue.

Firstly, using Eq. (6), one can calculate the mean square error of the level difference as,

$$s_h = s_{\Delta H_j}\sqrt{\frac{n}{2n_i(n-n_i)}} \tag{7}$$

For the gravity dam indicated in Fig. 2, the point most affected by deformation is $s_c = \pm 0.12mm$.

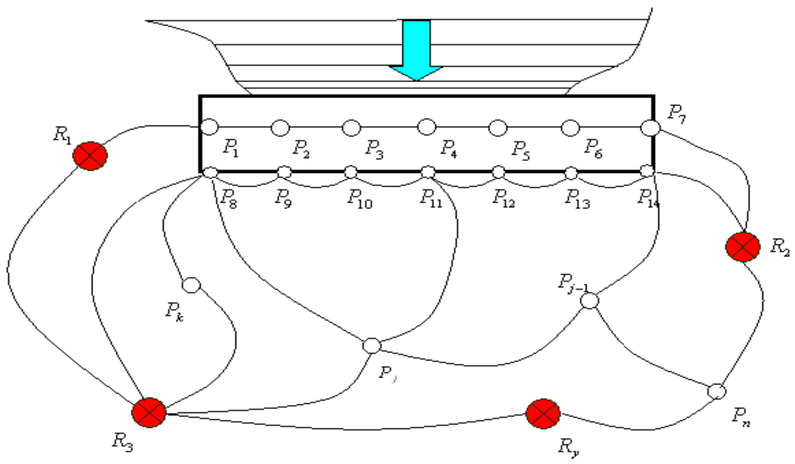

Fig. 2 Location of the fixed and control points – local reference network

Assuming that the required accuracy for the determination of the vertical displacement is $s_{\Delta H_5} = \pm 0.3mm$, for a total number of stations, $n = 20$ stations, and a total number of stations to point P_4, $n_5 = 10$ stations, the magnitude of the error of the level difference is

$$s_h = \pm 0.3\sqrt{\frac{20}{2\cdot 10(20-10)}} = \pm 0.094 \approx \pm 0.10mm$$

Therefore, in order to obtain the vertical displacement in the point most affected by displacement, P_4, the level difference in the station point must be determined with an accuracy of less than $\pm 0.10mm$. Thus, the instruments used for conducting measurements must fulfill certain conditions in order to ensure this required accuracy. For instance, in the case of a Zeiss Ni007 level, used with a 20 m span, based on the instrument error, one can obtain the mean square error of a reading on the invar rod.

The level difference between two control points calculated as a function of the readings on the scale of the rod, with the first horizon of the level, is

$$h_{i,1}^{I} = L_{0,1}^{(j-1)} - L_{0,1}^{(i)} \tag{8}$$

For this case, the mean square error is

$$s_h^1 = s_c\sqrt{2} = \pm 0.17mm \tag{9}$$

Similarly, the level difference obtained with the second horizon of the instrument will be,

$$h_{i,1}^{II} = L_{0,2}^{(j-1)} - L_{0,2}^{(i)} \tag{10}$$

and will have the same mean square error as expressed by Eq. (9). The average level difference will be expressed then as,

$$h_i^I = \frac{1}{2}(h_{i,1}^I + h_{i,2}^I) \tag{11}$$

The mean square error of the average level difference will then be,

$$s_h^{''} = \pm\frac{1}{\sqrt{2}}m_h^{'} = s_c = 0.12mm \tag{12}$$

and will therefore be equal to the reading error on the rod.

Since the mean square error of the level difference measured with one horizon of the instrument is larger then the one established "a priori" by calculation, ($\pm 0.12mm > \pm 0.10mm$), in order to ensure the required accuracy, one must also use the instrument using a second horizon. Consequently, the level difference will be calculated as the arithmetic mean of the values determined with the two horizons of the instrument.

$$h_i = \frac{1}{2}(h_i^I + h_i^{II}) \tag{13}$$

The associated mean square error of the average level difference will then be,

$$s_{h_i} = \pm\frac{1}{\sqrt{2}}s_h^{''} = \pm\frac{1}{\sqrt{2}}s_c = \pm 0.08mm \tag{14}$$

This error is smaller than the one established "a priori" by calculation, ($\pm 0.18mm < \pm 0.10mm$), so that the level difference of the most affected point will be determined with a required precision of $\pm 0.3mm$.

Finally, in order to ensure the required accuracy in each mid-point station, the following differences are accepted:

(a) The difference between the magnitudes of the level differences calculated using the readings on the two scales of the rod for one horizon of the instrument,

$$d_1 = h_{base}^I - h_{\sup l}^I \tag{15}$$

should not be obtained with an error superior to the following,

$$s_{d_1} = \pm\sqrt{\left(s_c\sqrt{2}\right)^2 + \left(s_c\sqrt{2}\right)^2} = \pm 2s_c \approx 0.25mm \tag{16}$$

Considering that one division of the micrometer is $0.05mm$, the minimum accepted accuracy corresponds to up to five micrometer divisions ($0.25/5 = 5 divisions$).

(b) The difference between the magnitudes of the level differences calculated using the both horizons of the instrument,

$$d_2 = h^I - h^{\Pi} \tag{17}$$

should not be obtained with an error larger than

$$s_{d_2} = \pm\sqrt{s_c^{\ 2} + s_c^{\ 2}} = \pm s_c\sqrt{2} \approx \pm 0.20mm \tag{18}$$

This error corresponds to three micrometer divisions.

Finally, for values calculated in both (a) and (b), if differences are greater then the calculated ones, measurements for the given station point must be performed again.

3. Algorithm for the Calculation of Vertical Deformations and Displacements Directly From the Variations of the Cyclically Measured Level Differences

Consider the high precision geometrical leveling shown in Fig. 1. First, the provisional level differences from the initial (zero) and current cycle are calculated in the form of arithmetic means of the values obtained with the two horizons of the instrument as follows.

$$h_i^0 = \frac{1}{2}(h_i^{0,I} + h_i^{0,\Pi})\ ,\ h_i^t = \frac{1}{2}(h_i^{t,I} + h_i^{t,\Pi})\ ,\ i = \overline{1,n}\ ,\ t = \overline{1,N} \tag{19}$$

During the initial cycle of measurements, the elevation of the fixed point, R_2, is calculated based on the elevation of the fixed point, R_1, and on the provisional level differences as,

$$\overline{H}_{R_2} = H_{R_1} + \sum_{i=1}^{n} h_i^0 \tag{20}$$

Theoretically, the above calculated level difference should be equal to the known – already calculated during the setting out of the leveling network – elevation, H_{R_2}. However, due to the unavoidable errors affecting the measurement process, a difference will occur between the two values, $\overline{H}_{R_2}$ and H_{R_2}. This difference is known as the closing error on elevations:

$$f_{h_0} = \overline{H}_{R_2} - H_{R_2} = \sum_{i=1}^{n} h_i^0 - (H_{R_2} - H_{R_1}) \tag{21}$$

Similarly, one can write the closing error on coordinates for the current cycle

$$f_{h_t} = \overline{\overline{H_{R_2}}} - H_{R_1} = \sum_{i=1}^{n} h_i^t - (H_{R_2} - H_1) \tag{22}$$

The unitary corrections for the two measuring cycles can be further calculated using the previously determined closing errors,

$$R_{h_0} = -\frac{f_{h_0}}{\sum_{i=1}^{n} D_i},\ R_{h_t} = -\frac{f_{h_t}}{\sum_{i=1}^{n} D_i} \tag{23}$$

where D_i are the magnitudes of the aiming distances.

The compensated elevations of the control point, j, obtained from the two cycles of measurements are then,

$$H_j^0 = H_{R_1} + \sum_{i=1}^{n} (h_i^0 + k_{h_0} D_i) \tag{24}$$

$$H_j^t = H_{R_1} + \sum_{i=1}^{n} (h_i^t + k_{h_t} D_i) \tag{25}$$

where: $i = \overline{1,n}$, $j = \overline{1,n-1}$, $t = \overline{1,N}$.

The magnitude of the vertical displacement for the considered control point can be expressed based on the elements measured during the two measurement cycles as follows,

$$\Delta H_j^t = \sum_{i=1}^{j} (h_i^t - h_i^0) - \sum_{i=1}^{j} D_i \left[\frac{\sum_{i=1}^{n} (h_i^t - h_i^0)}{\sum_{i=1}^{n} D_i} \right] \tag{26}$$

where the squared parentheses from the second right-hand member is a constant, K. Hence, the final formula for the calculation of the magnitude of the vertical displacement, a function of the elements measured during the two measurement cycles, becomes,

$$\Delta H_j^t = \sum_{i=1}^{j} (h_i^t - h_i^0) - K \sum_{i=1}^{j} D_i \tag{27}$$

The second term in the right hand side member of Eq. (27) represents the correction that corresponds to the errors occurring from the two measurement cycles.

In order to control the compensation of the vertical displacements, the following expression is used:

$$\Delta H_{R_2}^t = \Delta H_n^t = \sum_{i=1}^{n} (h_i^t - h_i^0) - K \sum_{i=1}^{n} D_i = 0 \tag{28}$$

If accidentally, between the two cycles of measurements, the support (benchmark) points, R_1 and R_2, have suffered vertical displacements, then, the vertical displacement of the control point, j, will be,

$$\Delta H_j^t = \Delta h_1 + \sum_{i=1}^{j} (h_i^t - h_i^0) - K \sum_{i=1}^{n} D_i \tag{29}$$

and the control formula becomes,

$$\Delta H^t_{R_2} = \Delta H^t_n = \Delta h_1 + \sum_{i=1}^{n}(h^t_i - h^0_i) - K\sum_{i=1}^{n} D_i = \Delta h_2 \tag{30}$$

If the calculation is performed with a mini0-calculator, the determination of the vertical deformations and displacements can simply be made using the following relation,

$$\Delta H^t_j = \Delta H^t_{j-1} + (h^t_i - h^0_i) - KD_i \tag{31}$$

combined with the following control relation,

$$\Delta H_{R_2} = \Delta H_n = \Delta H_{n-1} + (h^t_n - h^0_n) - KD_n = 0 \tag{32}$$

In most cases, for each measurement cycle, one must calculate the partial vertical deformations and displacements between the current cycle, t, and the previous one, $t-1$. This is expressed as,

$$\Delta H_j = \Delta H^t_j - \Delta H^{t-1}_j \tag{33}$$

The analysis of the calculated values will shed light on the behavior of the studied construction - in terms of vertical displacements – between the two conjugated measurement cycles.

4. Mathematical Model for the Estimation of the Accuracy of the Results

The estimation of the accuracy of the results is performed using the mean square error method. For the case of a given control point, j (with $j = \overline{1, n-1}$), the mean square errors of the elevations between the initial and current measurement cycles can be expressed as:

$$s_{H^0_j} = \pm\frac{s_{h^0}}{\sqrt{p_i}} \;,\; s_{H^t_j} = \pm\frac{s_{h^t}}{\sqrt{p_i}} \tag{34}$$

where the mean square of the level differences measured between the two cycles are,

$$s_{h^0} = \pm\frac{1}{2}\sqrt{\frac{1}{n}\sum_{i=1}^{n} d^0_i} \;,\; s_{h^t} = \pm\frac{1}{2}\sqrt{\frac{1}{n}\sum_{i=1}^{n} d^t_i} \tag{35}$$

In Eq. (35), d^0_i and d^t_i represent the intervals between the relative elevation differences determined with two horizons of the instrument, in each station point:

$$d_{0i} = h^{0,I}_i - h^{0,II}_i \;,\; d_{ti} = h^{t,I}_i - h^{t,II}_i \tag{36}$$

The weights of the control points, which are the same for all cycles of measurements, are expressed as,

$$p_j = \frac{n}{n_j(n-n_j)} \tag{37}$$

where: n is the total number of station points of the leveling; n_j represents the number of station points between the initial support point and the considered control point.

Using the means square errors expressed by Eq. (34), one can calculate the mean square error of the vertical displacement and deformations of the control points. For point j, this is expressed as,

$$s_{\Delta H^t_j} = \pm\sqrt{s^2_{H^t_j} + s^2_{H^0_j}} \tag{38}$$

Based on Eqs. (34), (35) and (37), one can obtain the mean square error of the vertical deformation of the point as follows:

$$s_{\Delta H^t_j} = \pm\frac{1}{2n}\sqrt{n_j(n-n_j)\sum_{i=1}^{n}(d^2_{0i} + d^2_{ti})} \tag{39}$$

Using the following constant,

$$K_1 = \frac{1}{2n}\sqrt{\sum_{i=1}^{n}(d_{0i}^2 + d_{ti}^2)} \tag{40}$$

one can write a direct formula for the calculation of the mean square error of the vertical displacement and deformation as,

$$s_{\Delta H_j^t} = \pm K_1\sqrt{n_j(n - n_j)} \tag{41}$$

Finally, the confidence interval within whose limits the true magnitude of the vertical deformation of point j will be found can be expressed as,

$$\Delta H_j^t - s_{\Delta H_j^t} \le \overline{\Delta H_j^t} \le \Delta H_j^t \pm s_{\Delta H_j^t} \tag{42}$$

5. Case Study

The present case study deals with the calculation of the vertical deformation and displacement of control points positioned at the base of a construction and the estimation of the accuracy of the results. Tables 1 and 2 contain the field measurement data, the calculation of the average level differences and of the intervals from each station point for each of the two measurement cycles. Table 3 contains the calculation and compensation of the vertical deformations and displacements of the control points of the high precision geometrical leveling supported on fixed points of known elevations. Table 3 also contains the mean square errors of the deformations from each control point.

Table 1. Initial measurement cycle

Station point	Length of aim D_i (m)	Instrument horizon $h_i^{0,I}$ (mm)	$h_i^{0,II}$ (mm)	Average elevation difference h_i^0	Interval d_{0i}
1	29	322.54	322.74	322.64	-0.20
2	20	17.88	17.70	17.79	0.18
3	22	22.53	22.31	22.42	0.22
4	18	11.16	11.36	11.26	-0.20
5	32	-29.69	29.45	-29.54	-0.24
6	20	30.33	30.23	20.28	0.10
7	19	-107.75	-107.62	-107.68	-0.13
8	23	-39.52	-39.40	-39.46	-0.12
9	31	343.34	343.50	343.42	-0.16

Table 2. Current measurement cycle

Station point	Length of aim D_i (m)	Instrument horizon $h_i^{t,I}$ (mm)	$h_i^{t,II}$ (mm)	Average elevation difference h_i^t	Exact d_{ti}
1	29	322.14	322.34	322.24	-0.20
2	20	17.55	17.75	17.65	-0.20
3	22	22.05	22.35	22.20	-0.20
4	18	11.70	11.60	11.65	0.10
5	32	-29.55	-29.75	-29.65	0.20
6	20	30.97	31.13	31.05	-0.16
7	19	-107.5	-107.32	-107.41	-0.18
8	23	-39.46	-39.34	-39.40	-0.12
9	31	344.66	344.36	344.51	0.30

Table 3. Calculation of the vertical deformation and of the mean square error

Station point	Level differences		$h_i^t - h_i^0$ (mm)	K^t (mm/m)	Length of aim D_i (m)	$-K\sum_{i=1}^{j} D_i$ or, $-KD_i$	Vertical deform. ΔH_j^t (mm)	Mean square error ΔH_j^t
	h_i^0 (mm)	h_i^t (mm)						
1	322.64	322.24	-0.40	0.0081	29	-0.24	-0.64	±0.12
2	17.79	17.65	-0.14		20	-0.16	-0.94	±0.16
3	22.42	22.20	-0.22		22	-0.18	-1.34	±0.18
4	11.26	11.65	0.39		18	-0.15	-1.09	±0.19
5	-29.57	-29.65	-0.08		32	-0.26	-1.43	±0.19
6	30.28	31.05	0.77		20	-0.16	-0.83	±0.18
7	-107.68	107.41	0.27		19	-0.15	-0.71	±0.16
8	-39.46	39.40	0.06		23	-0.19	-0.84	±0.12
9	343.42	344.51	1.09		31	-0.25	0.00	-

6. Conclusions

Following the present analysis and study, several conclusions are presented below.

1. The proposed procedure allows to “a priori” establish the appropriate level of accuracy required by field measurements in the case of the high-precision geometric leveling. This procedure is very useful in obtaining the most probable values of the vertical deformations and displacements of constructions. The procedure contributes to the analysis of the state of deformation of structures during both the construction phase and during service.
2. Comparing the pre- and post-calculated magnitudes of the measuring errors of the vertical displacements and deformations, one can verify the correctness of the preliminary study of the required accuracy for field measurements in all measurement cycles.
3. The proposed algorithm allows the fast and precise determination of the magnitude of vertical displacements and deformations in each point of interest of a construction. This is done by employing cyclical measurements of level differences measured using high-precision geodetic leveling method.
4. The proposed mathematical model allows the calculation of the mean square errors of the vertical displacements and deformations for each control point, based on the leveling measurements conducted in the field. The results are comparable to the ones obtained using rigorous methods such as the least squares method.

7. References

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