Solving System of Differential Equations with Polinomial Matrices

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Abstract: *The matrix calculus is widely applied nowadays in various branches of mathematics, mechanics, theoretical physics and geodesy. In this paper we will connect the matrix theory with system of differential equations. Firstly we will discuss, systems of first order homogeneous linear differential equations with constant coefficients, function of matrices, and we will give an example in this topic. Secondly we will discuss systems of homogeneous differential equation of order n with constant coefficients. The polynomial matrix, or λ matrix, is a special rectangular matrix, whose elements are polynomials in λ. If we use the theory of polynomial matrices, we can solve system of higher order homogeneous linear differential equations with constant coefficients. This problem have not solved yet. Nowadays we meet some very important problems, where we have to use this theory.*

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1. Systems of Linear Differential Equations with Constant Coefficients

1.1. Systems of first order homogeneous linear differential equations with constant coefficients

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ &. \\ &. \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{aligned}$$

where t is the variable, $x_1, x_2, \ldots, x_n$ are unknown functions of t, and $a_{ik}(i,k = 1,2,\ldots n)$ are complex numbers.

Setting $x(t) = \begin{bmatrix} x_1(t) \\ . \\ . \\ . \\ x_n(t) \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \\ \\ \\ \dot{x}_n \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & . & . & . & a_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & . & . & . & a_{nn} \end{bmatrix}$

the system can be written in the form of a single matrix differential equation $\dot{x} = Ax$.

The solution is: $x(t) = e^{At}C$, where e^{At} is defined by series $e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k$

Remark:

Let A be a square matrix and $f(\lambda)$ a function of a scalar argument λ. We wish to extend the function $f(\lambda)$ to a matrix value of the argument. We can use the "fundamental formula" as well.

Example*

$$\begin{aligned} \dot{x}_1 &= 3x_1 - x_2 + x_3 \\ \dot{x}_2 &= 2x_1 \qquad + x_3 \\ \dot{x}_3 &= x_1 - x_2 + 2x_3 \end{aligned}$$

The coefficient matrix is $A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristic polynomial is: $k(\lambda) = (\lambda - 1)(\lambda - 2)^2$

The fundamental formula is $f(A) = f(1)X + f(2)Y + f'(2)Z$, where, $X, Y, Z,$ are unknown matrices.

For $f(\lambda)$ we choose in succession $1, \lambda - 2, (\lambda - 2)^2$. We obtain:

$$\begin{aligned} &E = X + Y \\ &A - 2E = -X + Z \\ &(A - 2E)^2 = X \end{aligned}$$

Hence we determine X, Y, Z and substitute in the fundamental formula:

$$f(A) = f(1)\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} + f(2)\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} + f'(2)\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we now replace $f(\lambda)$ by $e^{\lambda t}$ we obtain:

$$e^{At} = e^t\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} (1+t)e^{2t} & -te^{2t} & te^{2t} \\ -e^t + (1+t)e^{2t} & e^t - te^{2t} & te^{2t} \\ -e^t + e^{2t} & e^t - e^{2t} & e^{2t} \end{bmatrix}$$

Thus

$$\begin{aligned} x_1 &= C_1(1+t)e^{2t} - C_2te^{2t} + C_3te^{2t} \\ x_2 &= C_1\left[-e^t + (1+t)e^{2t}\right] + C_2\left(e^t - te^{2t}\right) + C_3te^{2t} \\ x_3 &= C_1\left(-e^t + e^{2t}\right) + C_2\left(e^t - e^{2t}\right) + C_3e^{2t} \end{aligned}$$

where C_1, C_2, C_3 are constant.

*See Gantmacher: The Theory of Matrices Chapter V. Function of matrices §5.

1.2. Systems of homogeneous differential equation of order n with constant coefficients

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_1 x^{(1)} + a_0 x = 0$$

where *t* is the variable, *x* is an unknown function of *t* and $a_1, \ldots, a_n$ are complex numbers.

These systems have already been solved by Euler.

1.3. Systems of higher order homogeneous linear differential equations with constant coefficients

The polynomial matrix, or λ matrix is a special type of matrices which can be used to solve systems.

Definition1:

The polynomial matrix, or λ matrix is a matrix $A(\lambda)$ whose elements are polynomials of λ, $A(\lambda)=[a_{ik}(\lambda)]=[a_{ik}^{(0)}\lambda^l + a_{ik}^{(1)}\lambda^{l-1} + ... + a_{ik}^{(l)}]$ $(i=1,2,...,m;\ \ k=1,2,...,n)$, where l denotes the degree of polynomial $a_{ik}(\lambda)$.

Setting $A_j = [a_{ik}^j]$ $(i=1,2,...,m;\ k=1,2,...,n;\ j=0,1,....,l)$ we may represent the polynomial matrix $A(\lambda)$ in the form of a polynomial of λ with matrix coefficients:

$$A(\lambda)= A_0\lambda^l + A_1\lambda^{l-1} + ... + A_{l-1}\lambda\ + A_l$$

We want to solve the system:

$$\begin{aligned}
&a_{11}(D)x_1 + a_{12}(D)x_2 + ... + a_{1n}(D)x_n = 0\\
&a_{21}(D)x_1 + a_{22}(D)x_2 + ... + a_{2n}(D)x_n = 0\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_{m1}(D)x_1 + a_{m2}(D)x_2 + ... + a_{mn}(D)x_n = 0
\end{aligned}$$

where $x_1, x_2,..., x_n$ are unknown functions of the variable *t,*

$a_{ik}(D) = a_{ik}^{(0)}D^{(l)} + a_{ik}^{(1)}D^{(l-1)} + ... + a_{ik}^{(l)}$ $(i=1,2,...,m;\ k=1,2,...,n)$ is a polynomial of D with constant coefficients and $D=\dfrac{d}{dt}$ is the differential operator.

Setting $A(D):=[a_{ik}(D)]$ which is a polynomial matrix, the system has the form $A(D)x=0$.

If the matrix $A(D)$ is replaced by a canonical diagonal matrix in the system of differential equations, a reduced system of equations is obtained.

2. Elementary transformation of a Polynomial Matrix

This chapter includes basic knowledge of elementary transformations of polynomial matrices, equivalence of polynomial matrices and canonical diagonal form of matrices.

We introduce the following **elementary operations** on a polynomial matrix $A(\lambda)$:

2.1. Multiplication of any row by a non zero number

Multiplication of the i-th row by $c \neq 0$ is equivalent to the multiplication of the polynomial matrix on the left by the following square matrix S' of order m.

$$S' = \begin{array}{c} \begin{array}{ccccc} & & (i) & & \end{array} \\ \begin{bmatrix} 1 & . & . & . & 0\\ . & 1 & . & . & .\\ . & . & c & . & .\\ . & . & . & . & .\\ 0 & . & . & . & 1 \end{bmatrix} \end{array}$$

2.2. Addition of any row multiplied by an arbitrary polynomial of A to an other row

For the addition of the j-th row multiplied by $b(\lambda)$to the i-th row the matrix is

$$S'' = \begin{array}{cccccc} & & (i) & & (j) & \\ \end{array}$$
$$S'' = \begin{bmatrix} 1 & . & . & . & . & 0 \\ . & . & 1 & . & b(\lambda) & . \\ . & & & & & \\ . & & & & & \\ 0 & . & . & . & . & 1 \end{bmatrix}$$

2.3. Interchange of any two rows

$$\begin{array}{ccccccc} & & (i) & & (j) & & \end{array}$$
$$S''' = \begin{bmatrix} 1 & . & . & . & . & . & 0 \\ . & . & 0 & . & 1 & . & . \\ . & . & . & . & . & . & . \\ . & . & 1 & . & 0 & . & . \\ 0 & . & . & . & . & . & 1 \end{bmatrix}$$

Applying the operations 1, 2, or 3 the matrix $A(\lambda)$ is transformed to $S'A(\lambda)$, $S''A(\lambda)$, and $S'''A(\lambda)$, respectively. These operations are called **elementary left operation.**

In a similar way we define the **elementary right operations** on a polynomial matrix (these are performed not on the rows, but on the columns). The corresponding matrices (of order *n*) are:

$$T' = \begin{bmatrix} 1 & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & c & & . \\ . & . & . & . & . & . & . \\ 0. & . & . & . & . & . & 1 \end{bmatrix} \begin{array}{c} \\ \\ (i) \\ \\ \\ \end{array} \quad T'' = \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & & \\ . & b(\lambda) & . & . \\ 0 & . & . & . \end{bmatrix} \quad T''' = \begin{bmatrix} 1. & .0 \\ .. & 0. \\ .. & .. \\ .. & 1. \\ 0. & .1 \end{bmatrix} \begin{bmatrix} . & . & . & . & 0 \\ (i)1. & . & . & . & . \\ . & . & . & . & . \\ (j)0. & . & . & . & . \\ . & . & . & . & 1 \end{bmatrix} \begin{array}{c} \\ (i) \\ \\ (j) \\ \\ \end{array}$$

Remarks

In the matrices $S', S'', S''', T', T'', T'''$ all the elements not shown are 1 in the main diagonal and 0 elsewhere.

The determinant of every elementary matrix does not depend on λ, and is different from zero. Therefore each elementary left or right operation has an inverse operation which is also an elementary left or right operation.

Definition 2:

Two polynomial matrices $A(\lambda)$ and $B(\lambda)$ are called left equivalent, right equivalent, or equivalent if one of them can be obtained from the other by applying elementary left, elementary right, or elementary left and right operations.

Let $B(\lambda)$ be obtained from $A(\lambda)$ applying the left elementary operations corresponding to $S_1, S_2, \dots S_p$. Then $B(\lambda) = S_p S_{p-1} \dots S_1 A(\lambda)$. Denoting the product $S_p S_{p-1} \dots S_1$ by $P(\lambda)$ we have $B(\lambda) = P(\lambda) A(\lambda)$, where $P(\lambda)$, like each of the matrices $S_1, S_2, \dots S_p$ has a constant non zero (independent of λ) determinant.

Similarly, $B(\lambda) = A(\lambda) Q(\lambda)$, and in the case of (two sided) equivalence the equation $B(\lambda) = P(\lambda) A(\lambda) Q(\lambda)$ is valid. Here again $P(\lambda)$ and $Q(\lambda)$ are matrices with constant non-zero determinant independent of λ.

Definition 3:
A polynomial matrix is called a **canonical diagonal matrix** if it has the form

$$\begin{bmatrix} a_1(\lambda) & 0 & . & 0 & 0 & . & 0 \\ 0 & a_2(\lambda) & . & 0 & 0 & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & a_s(\lambda) & 0 & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & 0 & 0 \end{bmatrix}$$

where

a) the polynomials $a_1(\lambda), a_2(\lambda),..., a_s(\lambda)$ are not identically zero,
b) each of the polynomials $a_2(\lambda),..., a_s(\lambda)$ is divisible by the preceding,
c) the main coefficients of all the polynomials $a_1(\lambda), a_2(\lambda),..., a_s(\lambda)$ are equal to one.

We can prove that an arbitrary rectangular polynomial matrix $A(\lambda)$ is equivalent to a canonical diagonal matrix.

The equivalence is shown by constructing a sequence of elementary operations that give $P(\lambda)$ and $Q(\lambda)$. This algorithm is similar to Gauss's elimination method, and is used by well known programs (e.g. MATLAB).

3. Solving the systems of higher order homogeneous linear differential equations with constant coefficients with polynomial matrices

Let us consider the differential equation $A(D)x = 0$ again.

Let $P(D)$ and $Q(D)$ be matrices with constants non-zero determinants independent of D, such that they transform $A(D)$ into its canonical diagonal form $B(D) := P(D)A(D)Q(D)$

Multiplying the differential equation by $P(D)$ from left, then we obtain $P(D)A(D)x = 0$

On the other hand $Q(D)Q^{-1}(D) = I$ where I is unit matrix, hence $P(D)A(D)Q(D)Q^{-1}(D)x = 0$.

For $y := Q^{-1}(D)x$ and $B(D) := P(D)A(D)Q(D)$ we obtain $B(D)y = 0$, where $B(D)$ is a canonical diagonal matrix.

In this way we have transformed the original system to a system of higher order homogeneous differential equations with constant coefficients (type 2 of paragraph 1), which can be solved.

Finally $x = Q(D)y$.

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