COMBINING GNSS AND TERRESTRIAL OBSERVATIONS IN 2D GEODETIC NETWORKS THROUGH SEQUENTIAL ADJUSTMENT

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***Abstract**: With the advent of GNSS technology in geodesy, mathematical models have been developed to combine these observations with the terrestrial measurements. Most of these models are based on 3D coordinate systems. In this paper, a possibility of combining GNSS and terrestrial observations in two-dimensional geodetic networks is presented.*

***Keywords:** observations, network, sequential, adjustment*

1. Introduction

A sequential adjustment of a geodetic network assumes that measurements are processed in at least two stages. The results in the first stage will become observables in the next stage. This is particularly important for geodetic networks in which GNSS and classical observations have been made. As we know GNSS networks are three-dimensional networks. One will find a difficult task in combining both satellite and terrestrial observations, especially when terrestrial observations are only two-dimensional (horizontal directions and ellipsoidal distances). Of course a solution can be to express the GNSS baselines by his horizontal and vertical components and then to combine only the two-dimensional component of GNSS baselines with terrestrial 2D observations. This implies that GNSS measurements are reduced (modified) to be compatible with the mathematical model

In this paper is presented another method of combining GNSS and two-dimensional terrestrial data, without the need to reduce GNSS baselines to the ellipsoidal system. First GNSS baselines are adjusted in their own 3D coordinate system and then the results are converted in the ellipsoidal geodetic coordinate system. Next, through the sequential adjustment model, terrestrial observations are added, thus achieving the final results.

2. Least-squares adjustment with observation equations

Functional model

$$\mathbf{v} = \mathbf{A} \cdot \mathbf{x} + \mathbf{l} \tag{1}$$

Stochastic model

Stochastic model is given by the least-squares condition:

$$\mathbf{v}^{\mathbf{T}} \cdot \mathbf{P} \cdot \mathbf{v} \rightarrow \min \tag{2}$$

and by weight matrix below:

$$\mathbf{P} = \sigma_0^2 \cdot \mathbf{\Sigma_M} \tag{3}$$

Normal equation matrix

$$\mathbf{N} = \mathbf{A^T P A} \tag{4}$$

Estimated parameters

$$\mathbf{x} = \mathbf{-N^{-1} A^T P L} \tag{5}$$

A posteriori (estimated) standard deviation of unit weight:

$$S_0 = \pm\sqrt{\frac{\mathbf{v^T P v}}{n-h+d}} \tag{6}$$

Cofactor matrix for the unknowns:

$$\mathbf{Q_{xx}} = \mathbf{N^{-1}} \tag{7}$$

Estimated variance-covariance matrix for the unknowns:

$$\mathbf{\Sigma_{xx}} = S_0^2 \cdot \mathbf{Q_{xx}} \tag{8}$$

Sequential adjustment model

Sequential adjustment is a model which applies when observations are made in two stages, and there is a set of common parameters between the two stages [1], [2]. The parameters obtained in the first stage become observations in the second stage. Therefore, for every parameter, an equation like (9) should be drawn:

$$x^1 + v_x = x^0 + dx \tag{9}$$

or

$$v_x = dx + (x^0 - x^1) \tag{10}$$

The above equations elements are denoted as follows:

x^1 - estimated value of unknown x obtained in stage 1.

v_x - residual for unknown x.

x^0 - provisional value of the unknown x.

dx – differential shift of unknown x.

If we will denote:

$$l_x = x^0 - x^1 \tag{11}$$

then Eq. (10) becomes:

$$v_x = dx + l_x \tag{12}$$

Eq. (12) represents the functional model used to bring in the second stage, parameters obtained in first stage. Taking into account the observations made in the second stage the functional model for the sequential adjustment will be:

$$\begin{pmatrix} \mathbf{v_x} \\ \mathbf{v_2} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{pmatrix} + \begin{pmatrix} \mathbf{l_x} \\ \mathbf{l_2} \end{pmatrix} \tag{13}$$

The stochastic model is given by Eq. (14) and (15):

$$\mathbf{P_{xx}} = \sigma_0^2 \cdot \left(\mathbf{\Sigma^1_{xx}}\right)^{\mathbf{-1}} \tag{14}$$

and, the weight matrix which includes weights of the observations made in second stage:

$$\mathbf{P_s} = \begin{pmatrix} \mathbf{P_{xx}} & \mathbf{0} \\ \mathbf{0} & \mathbf{P_2} \end{pmatrix} \tag{15}$$

Subsequent computations are made with the same Eq. as in least-square adjustment with observation equations shown by the formulae (1)÷(8).

3. Coordinates and variance-covariance matrix conversion

Adjustment of a GNSS network, as we have said, is performed in a 3D coordinate system. When combining satellite observations with terrestrial 2D observations, a conversion between Cartesian coordinate system and ellipsoidal geodesic coordinate system is needed.

Coordinates conversion

Conversion between Cartesian geocentric coordinates and geodetic ellipsoidal coordinates will be realized with following Eq. [3]:

$$L = \text{arctg}\frac{Y}{X} \tag{16}$$

$$B_0 = \text{arctg}\frac{Z}{(1-e^2)(X^2+Y^2)^{1/2}} \tag{17}$$

Geodetic coordinates B and H^e are obtained iteratively applying Eq. (18), (19) and (20) until the desired precision is achieved:

$$N_i = a(1-e^2\sin^2 B_{i-1})^{-1/2} \tag{18}$$

$$H_i^e = \begin{cases} \dfrac{(X^2+Y^2)^{1/2}}{\cos B_{i-1}} - N_i, \text{ pentru } |B_0| < 45^0 \\ \dfrac{Z}{\sin B_{i-1}} - (1-e^2)N_i, \text{ pentru } |B_0| \geq 45^0 \end{cases} \tag{19}$$

$$B_i = \text{arctg}\left[\frac{Z}{(X^2+Y^2)^{1/2}} \cdot \frac{1}{1-\dfrac{e^2 N_i}{N_i + H_i^e}}\right] \tag{20}$$

Variance-covariance matrix conversion

Conversion of variance-covariance matrix between the two coordinates system is given in (21) [4].

$$\mathbf{\Sigma_{BLH}} = \left(\mathbf{R}\cdot\mathbf{H}^{-1}\right)\cdot\mathbf{\Sigma_{XYZ}}\cdot\left(\mathbf{R}\cdot\mathbf{H}^{-1}\right)^{\mathbf{T}} \tag{21}$$

where:

$$\mathbf{R} = \begin{pmatrix} -\sin B\cdot\cos L & -\sin B\cdot\sin L & \cos B \\ -\sin L & \cos L & 0 \\ \cos B\cdot\cos L & \cos B\cdot\sin L & \sin B \end{pmatrix}, \ \mathbf{H}^{-1} = \begin{pmatrix} \dfrac{1}{M} & 0 & 0 \\ 0 & \dfrac{1}{N\cos B} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{22}$$

Note that the unit measure for the elements in $\mathbf{\Sigma_{BLH}}$ matrix is radian.

4. 2D Adjustment on the reference ellipsoid

In this section, observations equations are presented for measurements of horizontal directions and distances. The coefficients of the equations are given for the ellipsoidal two-dimensional adjustment model [5].

Observation equation for directions reduced on the ellipsoid surface

$$\begin{aligned} v_{ij}^{\alpha} = &-dz_i + \frac{\sin A_{ji}^0\cdot M_j}{s_{ij}^0}dB_j - \frac{\cos A_{ji}^0\cdot N_j\cdot\cos B_j^0}{s_{ij}^0}dL_j + \frac{\sin A_{ij}^0\cdot M_i}{s_{ij}^0}dB_i - \\ &-\frac{\cos A_{ij}^0\cdot N_i\cdot\cos B_i^0}{s_{ij}^0}dL_i + \left(A_{ij}^0 - \alpha_{ij}^* - z_i^0\right) \end{aligned} \tag{23}$$

Weights of direction observations

Weights will be computed using the standard deviation of an observed direction (σ_α) and the *a priori* standard deviation of unit weight (σ_0):

$$p_\alpha = \frac{\sigma_0^2}{\sigma_\alpha^2} \tag{24}$$

Observation equation for distances reduced on the ellipsoid surface

$$\begin{aligned} v_{ij}^s = &-\cos A_{ji}^0 \cdot M_j \cdot dB_j - \sin A_{ji}^0 \cdot N_j \cdot \cos B_j^0 \cdot dL_j - \cos A_{ij}^0 \cdot M_i \cdot dB_i - \\ &-\sin A_{ij}^0 \cdot N_i \cdot \cos B_i^0 \cdot dL_i + \left(s_{ij}^0 - s_{ij}^* \right) \end{aligned} \tag{25}$$

Weights of distance observations

Standard deviation of a distance measured with total station is:

$$\sigma_s = a + b \cdot s[km] \tag{26}$$

Then, the weight of a distance equation will be:

$$p_s = \frac{\sigma_0^2}{\sigma_s^2} \tag{27}$$

Notations

In above equations following notations have been made:

M, N – principal radii of curvature;

A – geodetic azimuth;

α, s – horizontal direction and ellipsoidal distance;

B, L – geodetic coordinates;

z – orientation unknown;

5. Adjustment GNSS observations in Cartesian geocentric coordinate system

For every observed GNSS baseline a system of three equations like (28) must be drawn [6].

$$\begin{cases} v_{ij}^{\Delta X} = dX_j - dX_i + \left(X_j^0 - X_i^0 - \Delta X_{ij}^* \right) \\ v_{ij}^{\Delta Y} = dY_j - dY_i + \left(Y_j^0 - Y_i^0 - \Delta Y_{ij}^* \right) \\ v_{ij}^{\Delta Z} = dZ_j - dZ_i + \left(Z_j^0 - Z_i^0 - \Delta Z_{ij}^* \right) \end{cases} \tag{28}$$

Eq. (28) can be written in matrix form as:

$$\mathbf{v_{ij}} = \mathbf{A_{ij}} \cdot \mathbf{dX} + \mathbf{l_{ij}} \tag{29}$$

where:

X, Y, Z – geocentric Cartesian coordinates;

$$\mathbf{A_{ij}} = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{v_{ij}} = \begin{pmatrix} v_{ij}^{\Delta X} \\ v_{ij}^{\Delta Y} \\ v_{ij}^{\Delta Z} \end{pmatrix}, \ \mathbf{dX} = \begin{pmatrix} dX_i \\ dY_i \\ dZ_i \\ dX_j \\ dY_j \\ dZ_j \end{pmatrix}, \ \mathbf{l_{ij}} = \begin{pmatrix} X_j^0 - X_i^0 - \Delta X_{ij}^* \\ Y_j^0 - Y_i^0 - \Delta Y_{ij}^* \\ Z_j^0 - Z_i^0 - \Delta Z_{ij}^* \end{pmatrix} \tag{30}$$

Stochastic model for GNSS measurements

$$\mathbf{P} = \sigma_0^2 \cdot \mathbf{\Sigma_M^{-1}} \tag{31}$$

where:

$\mathbf{\Sigma_M}$ - variance-covariance matrix of GNSS measurements obtained after processing the GNSS baselines.

6. Case study

Based on the theoretical aspects presented so far, a case study on a geodetic network is presented. In the studied network satellite and terrestrial observations have been made. The measurements will be separated in two groups (satellite and terrestrial) and adjusted accordingly through sequential adjustment model. The chart of the process is shown in Fig. 1.

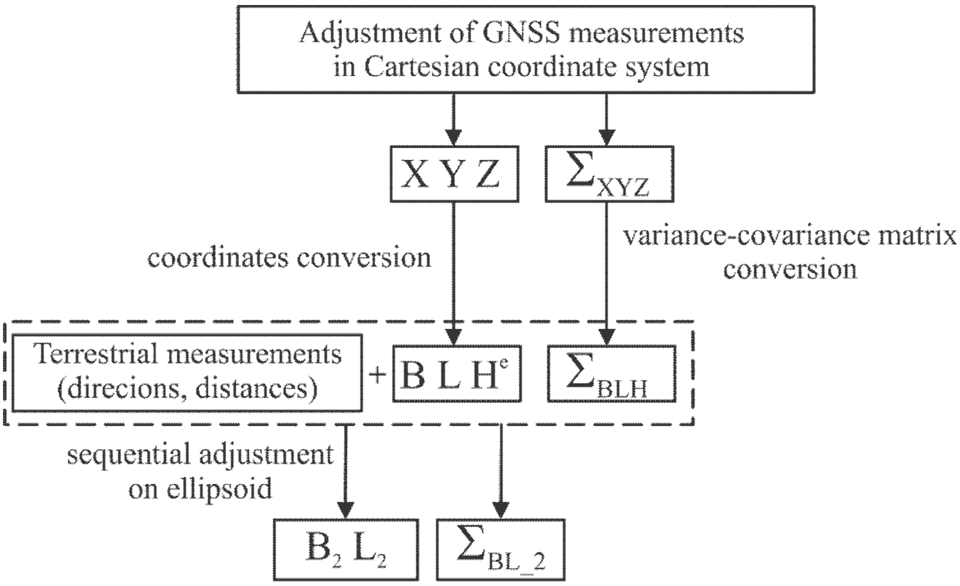


Fig. 1. Sequential adjustment of combined geodetic networks chart

GNSS network adjustment

In first stage GNSS measurements will be adjusted in 3D Cartesian coordinate system using the mathematical model shown in section 5. The reference system for the geodetic network is ETRS89. GNSS network characteristics are presented in Table 1 and GNSS observations are shown in Fig. 2.

Table 1. GNSS network characteristics

Reference system	ETRS89	Position unknowns	12
Coordinate system	Cartesian geocentric	GNSS baselines	6
Network dimension	3D	No. of observations	18
No. of points	5	Redundancy	6
No. of fixed points	1	Network type	minimally constrained
No. of free points	4	Rank-defect	0

from	to	meas	value	sigma_dx	sigma_dy	sigma_dz
1003	1002	dx	-232.3967	2.25e-06	3.96e-07	1.395e-06
1003	1002	dy	-76.9047	3.96e-07	6.4e-07	3.72e-07
1003	1002	dz	248.3185	1.395e-06	3.72e-07	2.25e-06
BALA	1002	dx	-4560.3331	7.84e-06	2.128e-06	5.9052e-06
BALA	1002	dy	4263.9776	2.128e-06	4e-06	1.998e-06
BALA	1002	dz	2123.4406	5.9052e-06	1.998e-06	1.369e-05
BALA	1003	dx	-4327.9322	1.225e-05	4.9105e-06	6.4015e-06
BALA	1003	dy	4340.883	4.9105e-06	5.29e-06	3.8502e-06
BALA	1003	dz	1875.1258	6.4015e-06	3.8502e-06	9.61e-06
1000	1001	dx	-29.0178	5.29e-06	2.576e-07	1.863e-06

Fig. 2. Excerpt from GNSS observations

Results of the adjustment are shown in Fig. 3 and the network plot in Fig. 5.

Point	Fix?	X_m________	Y_m________	Z_m________	Sx_cm	Sy_cm	Sz_cm	St_cm
BALA	yes	4062496.817	2119296.379	4422010.635	0	0	0	0
1000	no	4058136.304	2123546.525	4423959.084	0.35	0.13	0.27	0.46
1001	no	4058107.285	2123518.478	4423998.601	0.36	0.13	0.27	0.47
1002	no	4057936.485	2123560.357	4424134.077	0.28	0.19	0.31	0.46
1003	no	4058168.882	2123637.261	4423885.759	0.29	0.2	0.3	0.46

Fig. 3. GNSS network adjustments results in Cartesian coordinate system

Point	Coord	value	sigma_2_1	sigma_2_2
1000	L	27.6222178403339	4.75081668590967e-08	2.17223323641634e-08
1000	B	44.1984399102237	2.17223323641634e-08	3.84613574128518e-08
1001	L	27.6220752821537	3.31186177569285e-08	1.60178636997254e-08
1001	B	44.1989377762005	1.53938507960979e-08	2.99033239142325e-08
1002	L	27.6235301320713	0	0
1002	B	44.2006395005548	0	0
1003	L	27.6230345786371	0	0
1003	B	44.1975217725391	0	0

Fig. 4. GNSS network adjustments results in geodesic coordinate system

Because in the second stage the adjustment will be performed on the reference ellipsoid, results from the first stage will be converted from Cartesian Geocentric coordinate system to GRS 80 ellipsoidal coordinate system. This conversion is presented in section 3 Results are shown in Fig. 4.

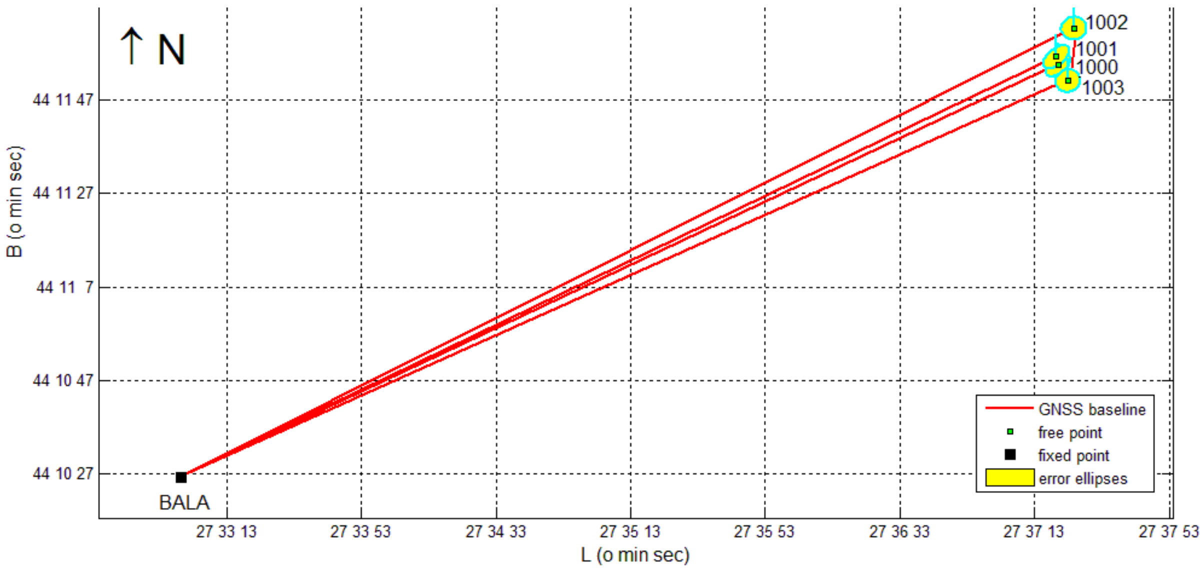


Fig. 5. GNSS network plot

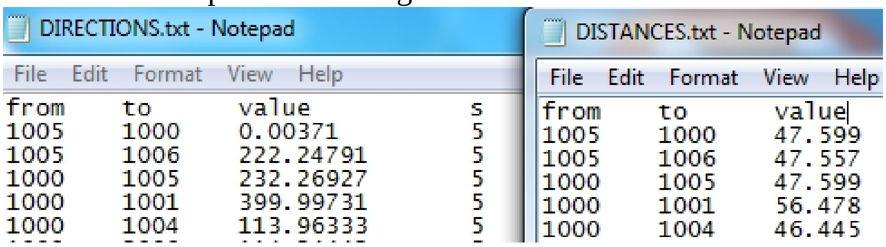
Sequential adjustment of combined geodetic network

The characteristics of the combined 2D geodetic network are shown in Table 2.

Table 2. Combined geodetic network characteristics (sequential adjustment)

Reference system	ETRS89	No. of stations	17
Coordinates system	Geodetic	No. of elements from previous adjustment	8
Ellipsoid	GRS80		
Network dimension	2D	No. of measurements	83
No. of points	20	Directions measured	39
No. of fixed points	0	Distances measured	44
No. of free points	20	Redundancy	34
Position unknowns	40	Rank-defect	0

Note that, because in the second stage the adjustment is performed in a two-dimensional coordinate system, only the 2D component from GNSS data was kept. The terrestrial measurements are presented in Fig. 6.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from	to	value	s
1005	1000	0.00371	5
1005	1006	222.24791	5
1000	1005	232.26927	5
1000	1001	399.99731	5
1000	1004	113.96333	5

DISTANCES.txt - Notepad
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from	to	value
1005	1000	47.599
1005	1006	47.557
1000	1005	47.599
1000	1001	56.478
1000	1004	46.445

Fig. 6. Excerpt from terrestrial observations files

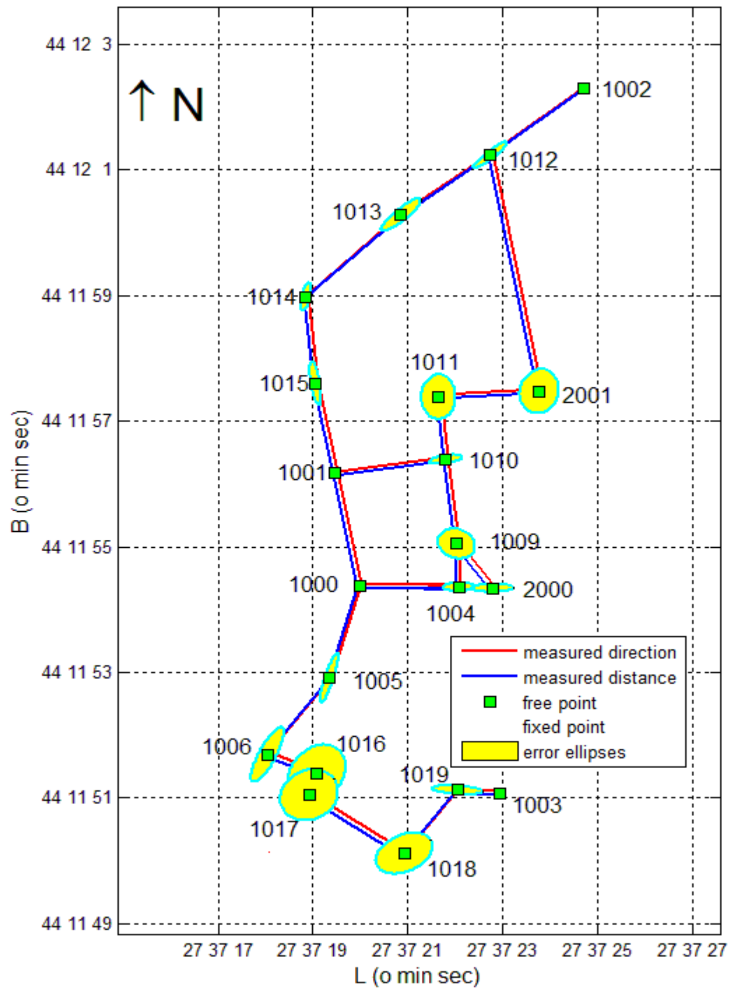

Fig. 7. Combined geodetic network plot

The adjustment of the combined geodetic network is performed using the sequential model, thus adding the terrestrial measurements to the GNSS network. The mathematical model for sequential adjustment is shown in section 2, while the observation equations and weights for terrestrial measurements are presented in section 4.

Final results for the combined geodetic network adjustment and the network plot are shown in Fig. 7, and respectively in Fig. 8.

Point	Fix?	B_grd______	L_grd______	Sb_cm	Sl_cm	St_cm	a_cm	b_cm	fi_gon
1000	no	44.19843991	27.62221785	0.2	0.2	0.3	0.2	0.1	36.49
1001	no	44.19893777	27.62207527	0.2	0.2	0.3	0.2	0.1	37.07
1002	no	44.2006395	27.62353014	0.2	0.2	0.3	0.2	0.2	0.73
1003	no	44.19752177	27.62303458	0.2	0.2	0.3	0.2	0.2	7.2
1005	no	44.19803136	27.62203984	1.2	0.4	1.3	1.3	0.2	19.44
1004	no	44.19843316	27.62279898	0.2	0.8	0.9	0.8	0.2	98.11
2000	no	44.19842838	27.62299364	0.3	1.0	1.0	1.0	0.3	98.86
1009	no	44.19862451	27.62278442	0.8	0.8	1.1	0.9	0.7	126.76
1010	no	44.19899715	27.62271945	0.2	0.8	0.9	0.8	0.2	90.37
1011	no	44.19927486	27.62267909	1.1	0.8	1.4	1.1	0.8	0.04
2001	no	44.19929871	27.62326829	1.1	0.9	1.4	1.1	0.9	14.33
1012	no	44.20034536	27.62298168	0.7	0.8	1.1	1.0	0.2	56.18
1013	no	44.20008012	27.62245956	0.8	0.9	1.3	1.2	0.4	53.54
1014	no	44.1997179	27.62190263	0.7	0.3	0.8	0.8	0.2	15.25
1015	no	44.1993344	27.62195935	1.1	0.3	1.1	1.1	0.2	191.24
1006	no	44.19769163	27.62167849	1.3	0.8	1.5	1.5	0.5	29.84
1016	no	44.19761057	27.62196865	1.4	1.3	1.9	1.5	1.2	45.61
1017	no	44.19751255	27.62192188	1.2	1.3	1.8	1.4	1.2	66.46
1018	no	44.19725435	27.6224777	1	1.4	1.7	1.4	0.9	71.79
1019	no	44.19753465	27.62278943	0.2	1.2	1.2	1.2	0.2	104.64

Fig. 8. Combined geodetic network – final results in geodetic coordinate system

7. Conclusions

Final results obtained using sequential adjustment model are expected to be the same as if measurements would have been processed together.

It is not necessary to now the configuration of the network adjusted in first stage, nor the type of the measurements involved. The only things relevant for the adjustment in the second stage are the parameter values and the variance-covariance matrix obtained in first stage.

Through the sequential model, precisions of the control points can be brought into the mathematical adjustment model.

The method presented in this paper is very useful to make the transition from three-dimensional positioning to two-dimensional positioning.

Even if control points are not assumed fixed in the second stage, the normal equation matrix isn't rank-deficient because the datum is fixed by the positions of points computed in the first stage and introduced as observations in the second stage.

In the end, if needed, the results in the geodetic ellipsoidal coordinate system can be converted or transformed in any map projection using the corresponding relation.

8. References

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