CALCULATION ALGORITHM FOR THE DETERMINATION OF VERTICAL DEFORMATIONS AND DISPLACEMENTS OF CONSTRUCTIONS, USING PRECISION TRIGONOMETRIC LEVELING

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Abstract: *The precision trigonometric leveling method used to determine vertical deformations and displacements of constructions implies carrying out cyclic measurements on the field of control points which are located on the analyzed structure with reference to a number of fixed points placed on solid, stable ground, outside the influence area of the analyzed structure.*

The method of precision trigonometric leveling is particularly useful when determining the vertical deformations and displacements of farther located control points or when determining the level differences between the fixed points and the control points is affected during field measurements by the measurement errors of the zenithal angles, as well by the atmospheric refraction coefficient.

This paper presents a new mathematical model for determining accurate values of zenith angles as well as the atmospheric refraction coefficients.

Keywords: *trigonometric leveling, elevation, deformation, zenithal angle, refraction.*

1. Introduction

Of outstanding importance in the study and analysis of the *in situ* behavior of constructions, in execution and in operation, are the data on deformations and their vertical displacements, called settlings for negative values and elevations for positive values.

The principle of determining vertical deformations and displacements lies in cyclical measuring of the control points heights, also called compaction marks or mobile landmarks, embedded on the studied construction in relation to a few fixed landmarks, of reference, situated on stable ground and outside the influence area of the construction. Determination of heights is achieved by precise geodetic measurements, using the geometric or trigonometric leveling method, depending on the actual conditions in the site.

The geodetic trigonometric leveling method, also called of precision, is used to determine the vertical deformations and displacements of the control points set on the studied building, particularly of the remote and hard to access points of tall buildings. Accuracy in determining the level differences of fixed landmarks outside the building and the building control points, is influenced by the errors of measuring zenithal angles and accurate knowledge of the atmospheric refraction coefficient during measurements.

The paper shows a mathematical model of treating excess measurements, by determining the most likely values of zenithal angles and of atmospheric refraction coefficients, in order to increase the accuracy of determining the control points heights, and hence of vertical deformations and displacements, using the forward / direct intersection. The points from which measurements are made are fixed points of the reference network, from which measurements are made for horizontal deformations, too.

Vertical atmospheric refraction occurs when zenithal angles measurement used in trigonometric leveling is performed. Using a mean refraction coefficient regardless of the period of measurements, will lead to uncertain results, with errors in establishing the control points heights.

Using the forward geodetic intersection method in determining the level differences between two or several fixed points outside the studied building and the control points on the building, provides a surplus of measurements required for compensation. Getting the most probable values is possible when determining the most probable values of zenithal angles and of atmospheric refraction coefficients, affecting the measured sizes for directions from each point of the station.

The paper presents a method for determining atmospheric refraction coefficients, also taking into account the measurement errors of zenithal angles, which reflect as accurate as possible the situation in the field, provided that all level differences are measured trigonometrically.

2. Presentation of calculation and compensation algorithm

The *A* and *B* fixed points outside the building and the control points $P_1, P_2, \dots, P_j, \dots, P_N$ set on the building, are considered. The $P_j\,(j=\overline{1,N})$ control points heights, will be determined using the forward leveling intersection method at the two station points, *A* and *B*. Horizontal distances, corresponding to visas will be determined either from the rectangular coordinates known from the reference microtriangulation network, as in the case of weight or arch dams tracking or by electronic measurement. Elements directly measured in each cycle are the zenithal angles of the visas, Z_i, where $i=\overline{1,n}$.

Because visas from station points to control points are unilateral, level differences between station points, *A* and *B*, and the control point P_j will be expressed by the relations (Fig. 1):

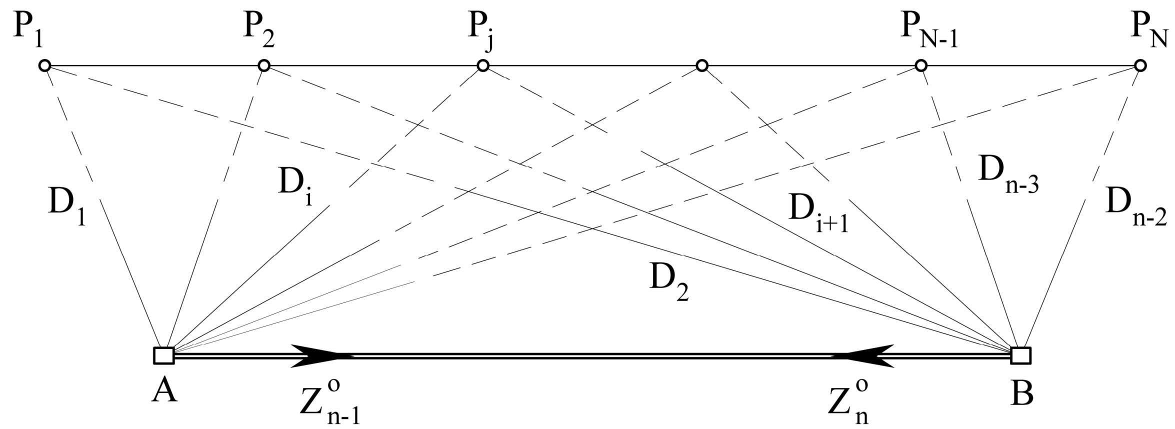

Fig. 1. Station points and control points positioning

$$h_{AP_j} = D_i \; ctg \; Z_i + I_A - S_j + C_i - C_i \, K_1 \; , \tag{1}$$

$$h_{BP_j} = D_{i+1} \; ctg \; Z_{i+1} + I_B - S_j + C_{i+1} - C_{i+1} K_2 \; , \tag{2}$$

where D_i and D_{i+1} are horizontal distances from the *A* and *B* station points to the control point P_j. Their sizes are calculated from the rectangular coordinates of local microtriangulation network points or electronic measurement; Z_i, Z_{1+1} are compensated zenithal angles sizes, unilaterally measured from *A* and *B* stations to P_j point; I_A and I_B are the unit heights from *A* and *B*; S_j is the sight signal height from P_j; K_1 and K_2 are sizes of the atmospheric refraction coefficients affecting the sizes of zenithal angles on the directions between the two stations, whose sizes are not known and have to be determined; C_i and C_{i+1} are corrections of the Earth curvature, expressed by the forms:

$$C_i = \frac{D_i^2}{2R} \; , \;\; C_{i+1} = \frac{D_{i+1}^2}{2R} \; , \;\; R \approx 6379 \; km. \tag{3}$$

Level differences (1) and (2), determined trigonometrically, are functions of the measured elements

$$h_{AB} = f(D, Z, K), \tag{4}$$

where the measurement errors of the instrument and signal height are neglected. In this way, the level difference error will be expressed by the relation:

$$s_h = \pm \left[s_D^2 \, ctgZ + \frac{D}{\sin^2 Z} \left(\frac{s_Z^{cc}}{\rho^{cc}} \right)^2 + \frac{D^4}{4R^2} s_K^2 \right]^{\frac{1}{2}} . \tag{5}$$

For the simplifying case, it is seen that the level difference error depends, particularly, on the accuracy of the atmospheric refraction coefficient knowledge. The condition to find out the atmospheric refraction coefficient *K* will result from the existence of measurements excess, so that the height of P_j point may be calculated from two independent determinations. Unlike the initial method, when it was considered that angle / zenithal distances are not affected by the inherent errors in the measurement [5], this algorithm will also take into consideration the influence of zenithal angles measurement errors. For this, it will be proceeded as follows:

From AP_jB triangle equality will result

$$h_{AP_j} + (-h_{BP_j}) = h_{AB} , \tag{6}$$

which is a condition equation, where h_{AP_j} *şi* h_{BP_j} are the compensated values of the level differences between the *A* and *B* station points and the P_j control point, unilaterally determined, using relations (1) and (2) while h_{AB} is the level difference between the station points, trigonometrically determined, with sight in both directions.

Introducing the relations (1) and (2) in (6), will lead to:

$$D_i \, ctg \; Z_i + I_A - S_{P_j} + C_i - C_i K_1 - D_{i+1} \, ctg \; Z_{i+1} - I_B + S_{P_j} - C_{i+1} + C_{i+1} K_2 = h_{AB} \; . \tag{7}$$

In the equation obtained, the K_1 and K_2 atmospheric refraction coefficients, corresponding to the observations from *A* and *B* station points will be replaced by a provisional value, calculated by the form:

$$K_0 = 1 - \frac{\left[\left(Z_{n-1}^0 - Z_n^0 \right) - 200^g \right]^{cc}}{\rho^{cc}} \cdot \frac{R}{D_{AB}} , \tag{8}$$

where: Z_{n-1}^0 and Z_n^0 and are the average values of mutually measured zenithal distances from *A* to *B* and from *B* to *A*; D_{AB} is the horizontal distance between the *A* and *B* station points, obtained either from the rectangular coordinates of the points, belonging to the microtriangulation network, e.g. from a dam, or by electronic measurement.

The expression (8) is to be applied within a restriction [6], which stems from the fact that the atmospheric refraction is a periodic function of twenty-four hours. Therefore, zenithal measurements should be made during the zenithal relatively stable period of the phenomenon, between the hours 10 a.m. and 3 p.m. of the day.

In equation (7), the compensated / probable values of zenithal angle, of atmospheric refraction coefficients and of the level difference between station points, will be replaced by the following relations:

$$Z_i = Z_i^0 + v_i,\ Z_{i+1} = Z_{i+1}^0 + v_{i+1},\ K_1 = K_0 + dK_1,\ K_2 = K_0 + dK_2,\ h_{AB} = h_{AB}^0 + dh, \quad (9)$$

where: Z_i^0 and Z_{i+1}^0 are average values of zenithal angles measured with a precision theodolite or with a performing total station; K_0 represents the provisional value of the atmospheric refraction coefficient, calculated by form (8); v_i and v_{i+1} are corrections applied to the average values of zenithal angles; dK_1 and dK_2 are (unknown) corrections that will be applied to the average atmospheric refraction coefficient, to obtain refractive coefficients corresponding to the two station points; h_{AB}^0 is the level difference calculated from the mutual of the *AB* side mutual sight. It will result:

$$\begin{aligned} &D_i\, ctg\,(Z_i^0 + v_i) - D_{i+1}\, ctg\,(Z_{i+1}^0 + v_{i+1}) + I_A + C_i - C_i(K_0 + dK_1) - \\ &- I_B - C_{i+1} + C_{i+1}(K_0 + dK_2) = h_{AB}^0 + dh, \text{ where } S_j = 0. \end{aligned} \quad (10)$$

Having a non-linear form, the principle of minimum could not be applied. Therefore, linearization will made bydevrlopment in Taylor series, retaining only the first order terms. Thus, a linear equation of condition will result:

$$\begin{aligned} &D_i\, ctg\,(Z_i^0 + \frac{v_i^{cc}}{\rho^{cc}\sin^2 Z_i^0}) - D_{i+1}\, ctg\,(Z_{i+1}^0 + \frac{v_{i+1}^{cc}}{\rho^{cc}\sin^2 Z_{i+1}^0}) + I_A - \\ &- I_B + C_i - C_{i+1} - C_i(K_0 + dK_1) + C_{i+1}(K_0 + dK_2) = h_{AB}^0 + dh, \end{aligned} \quad (11)$$

or

$$\frac{D_i}{\sin^2 Z_i^0}\cdot\frac{v_i^{cc}}{\rho^{cc}} - \frac{D_{i+1}}{\sin^2 Z_{i+1}^0}\cdot\frac{v_{i+1}^{cc}}{\rho^{cc}} - C_i K_1 + C_{i+1}K_2 - dh + w_j = 0, \quad (12)$$

where the free term of the equation of condition will be:

$$w_j = D_i ctgZ_i^0 - D_{i+1} ctgZ_{i+1}^0 + I_A - I_B + C_i - C_{i+1} - C_i K_1 + C_{i+1}K_2 - h_{AB}^0 \ . \quad (13)$$

The number of linear equations of condition (12) will be equal to the number of control points on the studied building.

Further on, using the mutual observations on the *AB* basic side, the equations can be written:

$$\begin{aligned} h_{AB} &= D_{AB} ctgZ_{n+1}^0 + I_A - S_B - C_A K_1, \\ -h_{BA} &= D_{BA} ctgZ_{n+2}^0 + I_B - S_A - C_B K_2 \end{aligned} \ . \quad (14)$$

The second equation is substracted from the first and it will result:

$$2h_{AB} = D_{AB} ctgZ_{n-1}^0 - D_{BA} ctgZ_n^0 + I_A - I_B + S_A - S_B - C_A K_1 + C_B K_2 \quad (15)$$

and introducing notations

$$h_{AB} = h_{AB}^0 + dh,\ K = K_0 + dK,\ Z_{n-1} = Z_{n-1}^0 + v_{n-1},\ Z_n = Z_n^0 + v_n, \quad (16)$$

after linearization the equation will result:

$$\frac{D_{AB}}{\sin^2 Z_{n-1}^0}\cdot\frac{v_{n+1}^{cc}}{\rho^{cc}} - \frac{D_{BA}}{\sin^2 Z_n^0}\cdot\frac{v_{n+2}^{cc}}{\rho^{cc}} - C_A dK_1 + C_B dK_2 - 2dh + w_r = 0, \quad (17)$$

where the free term will be:

$$w_r = D_{AB} ctgZ_{n-1}^0 - D_{BA} ctgZ_n^0 - C_A K_0 + C_B K_0 - 2h_{AB}^0 . \quad (18)$$

Linear equation of condition (17) corresponds to the level difference between station points *A* and *B*.

Taking into account the equations (12) and (17), for the case of a *N* number of control points, determined by the forward leveling intersection, from two station points, a number of *r = N+1* equations of angular corrections and of additional unknowns will be given, having the general form

$$\begin{aligned} &a_1 v_1 + a_2 v_2 + A_1 dK_1 + B_1 dK_2 + C_1 dh + w_1 = 0, \\ &b_1 v_3 + b_2 v_4 + A_2 dK_1 + B_2 dK_2 + C_2 dh + w_2 = 0, \\ &-------------------------- \\ &r_1 v_{n-1} + r_2 v_n + A_r dK_1 + B_r dK_2 + C_r dh + w_r = 0 . \end{aligned} \quad (19)$$

Characteristic is the fact that the obtained system is a system of equations of condition, with three additional unknown dK_1, dK_2 *and* dh. Therefore, it will be called the system of condition equations with additional unknowns. Compensation in this case should be regarded as a particular case of the general problem, when the compensated values of the measured variables $X_1, X_2, \ldots, X_n$ are related by condition equations of general form

$$E_j : f_j(X_1, X_2, \ldots, X_n ; x, y, \ldots, u) = 0 , \quad (20)$$

where *j = a, b,…, r*, and *x, y,…,u* are additional unknowns, whose number, in the general case, is equal to *t*, as they are unmeasured magnitudes. If in equation (20) the compensated values are substituted

$$\begin{cases} X_i = M_i^0 + v_i \\ x = x_0 + dx,\ y = y_0 + dy,\ \ldots,\ u = u_0 + du , \end{cases} \quad (21)$$

the ecuations will be given:

$$E_j : f_j(M_1^0 + v_1, M_2^0 + v_2, \ldots, M_n^0 + v_n ; x_0 + dx, y_0 + dy, \ldots, u_0 + du) = 0 . \quad (22)$$

Appreciating corrections $v_i, i = \overline{1,n}$, as independent of M_i^0 measurement results, developing in the Taylor series and only limited to linear terms, the linear system of condition equations with additional unknowns will be obtained, similar to the expression (19):

$$\begin{cases} a_1 v_1 + a_2 v_2 + \ldots + a_n v_n + A_1 dx + B_1 dy + \ldots + T_1 du + w_1 = 0, \\ b_1 v_3 + b_2 v_4 + \ldots + b_n v_n + A_2 dx + B_2 dy + \ldots + T_2 du + w_2 = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \underbrace{r_1 v_1 + r_2 v_2 + \ldots + r_n v_n}_{n} + \underbrace{A_r dx + B_r dy + \ldots + T_r du}_{t} + w_r = 0 . \end{cases} \quad (23)$$

The number of equations being *r*, the question of compensation will make sense and will be made only if there are the inequalities

$$r > t \ \text{și} \ n > r - t . \quad (24)$$

There are several ways to solve. Of these, direct solving will be used, preserving both the v_i corrections and the *dx, dy,…, du* additional unknowns. In this case, without detailing the mathematical model, to the linear system of condition equations with additional unknowns, using the method correlates, the following system of normal equations, where the number of equations is equal to the number of unknowns (the $k_1, k_2, ..., k_r$ correlates and the *dx, dy,…, du* additional unknowns) will correspond:

$$\begin{cases} [aa]k_1 + [ab]k_2 + ... + [ar]k_r + A_1dx + B_1dy + ... + T_1du + w_1 = 0, \\ [ab]k_1 + [bb]k_2 + ... + [br]k_r + A_2dx + B_2dy + ... + T_2du + w_2 = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ [ar]k_1 + [br]k_2 + ... + [rr]k_r + A_rdx + B_rdy + ... + T_rdu + w_r = 0, \\ A_1k_1 + A_2k_2 + ... + A_rk_r + 0dx + 0dy + ... + 0du + 0 = 0, \\ B_1k_1 + B_2k_2 + ... + B_rk_r + 0dx + 0dy + ... + 0du + 0 = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ T_1k_1 + T_2k_2 + ... + T_rk_r + 0dx + 0dy + ... + 0du + 0 = 0. \end{cases} \quad (25)$$

For the analyzed study case, to the system (19) the normal system of corrections and additional unknowns will correspond:

$$\begin{cases} [aa]k_1 + [ab]k_2 + ... + [ar]k_r + A_1dK_1 + B_1dK_2 + C_1dh + w_1 = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ [ar]k_1 + [br]k_2 + ... + [rr]k_r + A_rdK_1 + B_rdK_2 + C_rdh + w_r = 0, \\ A_1k_1 + A_2k_2 + ... + A_rk_r + 0dK_1 + 0dK_2 + 0dh + 0 = 0, \\ B_1k_1 + B_2k_2 + ... + B_rk_r + 0dK_1 + 0dK_2 + 0dh + 0 = 0, \\ C_1k_1 + C_2k_2 + ... + C_rk_r + 0dK_1 + 0dK_2 + 0dh + 0 = 0. \end{cases} \quad (26)$$

By solving the normal equations, the *r* $k_1, k_2, ..., k_r$ correlated ones and the three values of the unknowns corrections $dK_1,\ dK_2\ and\ dh$ will be determined. The correlates will be replaced in the correction equations system, in terms of the correlates:

$$E_i : v_i = a_ik_1 + b_ik_2 + ... + r_ik_r\ ,\ i = \overline{1,n}\ . \quad (27)$$

Based on v_i corrections and corrections of the $dK_1,\ dK_2\ and\ dh$, unknowns, the most probable values of zenithal distances, of atmospheric refraction coefficients are obtained, corresponding to the two stations *A* and *B*, and the level difference between the two station points, with relations (16):

$$Z_i = Z_i^0 + v_i\ ,\ K_1 = K_0 + dK_1\ ,\ K_2 = K_0 + dK_2\ ,\ h_{AB} = h_{AB}^0 + dh\ . \quad (28)$$

Compensation control is done by replacing in the system of condition equations of corrections (15), the compensated values of measured sizes and additional unknowns, which must match within the precision of calculation.

Finally, firstly the compensated value of the level difference between station points is calculated, with the relation:

$$h_{AB} = h_{AB}^0 + dh\ . \quad (29)$$

In terms of the known height of the *A* station point, the *B* station point height is determined

$$H_B = H_A + h_{AB}\ , \quad (30)$$

after which, based on the compensated values of zenithal angles and of atmospheric refraction coefficients, the level difference between the *A* and *B* station points and the $P_j (j = \overline{1,N})$ control points are calculated with the forms (1) and (2), and hence their heights

$$H_{P_j}^{A} = H_A + h_{AP_j}\ ,\ H_{P_j}^{B} = H_B + h_{BP_j}\ . \tag{31}$$

By means of $H_{P_j}^{0}$ control point height corresponding to the initial / zero readings cycle, and of the height corresponding to the $t(t=\overline{1,T})$, $H_{P_j}^{t}$ cycle, the following data will be obtained:

a) Deformation and absolute vertical displacement of the $P_j(j=\overline{1,N})$ control point

$$\Delta H_{P_j}^{t} = H_{P_j}^{t} - H_{P_j}^{0}\ . \tag{32}$$

b) Deformation and the total vertical displacement produced between initial and final cycle cycle

$$\Delta H_{P_j}^{T} = H_{P_j}^{T} - H_{P_j}^{0}\ . \tag{33}$$

c) Deformation and partial vertical displacement produced in the interval between two cycles of combined measurements

$$\Delta H_{P_j} = H_{P_j}^{t} - H_{P_j}^{t-1}\ . \tag{34}$$

d) Deformation and average vertical displacement of the building, corresponding to the readings cycle

$$\Delta H_{med}^{t} = \frac{1}{N}(\Delta H_{P_1}^{t} + \Delta H_{P_2}^{t} + \ldots + \Delta H_{P_N}^{t})\ . \tag{35}$$

e) Deformation and vertical movement rate of a control point

$$v_{P_j} = \frac{\Delta H_{P_j}^{t}}{t}\ , \tag{36}$$

where v_{P_j} is the deformation and vertical movement rate; $\Delta H_{P_j}^{t}$ is the deformation and the vertical movement yelded at a time *t*, expressed in months or years.

f) Yelding rate of the mean vertical displacement and deformation of the entire structure

$$v_{med} = \frac{1}{N}\sum_{j=1}^{N} v_{P_j}\ . \tag{37}$$

All these elements serve to drafting the documentation on the vertical evolution of the studied construction.

3. Evaluation of compensation precision of measurements conditioned with additional unknowns

For a complete and fair assessment of the precision of the results, the following errors will be determined:

a) *Mean post-compensation square error* of elements measured in the field (zenithal distance) using the relation [4]

$$s_0 = \pm\sqrt{\frac{[vv]}{r-t}}\ , \tag{38}$$

where

$$[vv] = v_1^2 + v_2^2 + \ldots + v_n^2\ , \tag{39}$$

or, for control

$$[vv] = -k_1 w_1 - k_2 w_2 - ... - k_r w_r = -[kw] \, . \tag{40}$$

In the relation (38), the denominator represents the difference between the number of condition equations and number of additional unknowns.

b) *Mean square errors of the* dK_1, dK_2 *and* dh *additional unknowns*

For the general case of the *dx* unknown, in fact of the *x* compensated unknown, the error will be

$$s_x = \pm s_0 \sqrt{Q_{xx}} \, . \tag{41}$$

For the analyzed case, the errors estimating the atmospheric refraction coefficient, in case of zenithal measurements from two station points will be:

$$s_{k_1} = \pm s_0 \sqrt{Q_{k_1 k_1}} \, , \quad s_{k_2} = \pm s_0 \sqrt{Q_{k_2 k_2}} \, . \tag{42}$$

Mean square error of the level difference between the station points will be

$$s_h = \pm s_0 \sqrt{Q_{hh}} \, . \tag{43}$$

To calculate the unknowns weight coefficients, it is recommended to solve the system of normal equations by the undetermined coefficients method, by reversing the normal equation coefficients matrix. In this case, the weight coefficients for the three unknowns $Q_{k_1 k_1}$, $Q_{k_2 k_2}$ and Q_{hh}, will be placed on the main diagonal of the matrix of weight coefficients (co-factors).

4. Conclusions

The precision trigonometric leveling method, used to determine vertical deformations and displacements of the control points on the studied construction, provides comparable accuracy using precision geometric leveling, but in difficult conditions of the measurement.

This method provides the opportunity of compensating the vertical angles of the visas, as well as of atmospheric refraction coefficients for each station, while by the surplus measurements by the forward leveling intersection method provides the network compensation by the method of conditioned measurements with additional unknowns, having the advantage technological facilities of execution, as well as of rapidity.

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