STUDY OF DEFLECTION OF THE VERTICAL DETERMINATION METHODS AND THE INFLUENCE ON THE TRADITIONAL TERRESTRIAL THREE-DIMENSIONAL GEODETIC MEASUREMENTS

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Abstract: *In present Global Navigation Satellite Systems (GNSS) are extensively used for national geodetic networks development and surveying. However, there are some applications where GNSS observations have to be integrated with traditional three-dimensional terrestrial geodetic measurements (e.g., underground engineering surveying, hydro electrical construction monitoring etc.). In order to integrate the GNSS observations and traditional measurements using total stations, it is necessary to know the Deflection of the Vertical (DoV), defined as the angle between the gravity vector that coincide with plumb line and the ellipsoidal normal in a given point. This paper is focused on studies of DoV components influence on the traditional terrestrial three-dimensional geodetic measurement: distance, horizontal and vertical angles. DoV components were calculated in the test network using geodetic (ellipsoidal) height differences determined by GNSS measurements and normal height differences from levelling network. The preliminary results show the necessity to use I-II order levelling measurements and at least 1 hour GNSS observation in order to determine DoV components with a level of precision comparable to the angular accuracy achievable by 1 arcsec precision total station.*

Keywords: *Deflection of the Vertical, GNSS observations, traditional geodetic measurements, geodetic network, gravity vector, plumb line, ellipsoidal normal, levelling.*

1. Introduction

The worldwide ongoing process of using Global Navigation Satellite Systems (GNSS) for national geodetic networks development and surveying are related to International Terrestrial Reference System (ITRS). However, there are some applications where GNSS observations have to be integrated with traditional three-dimensional terrestrial geodetic measurements using total stations that are related to astronomical topocentric system of coordinates (e.g., underground engineering surveying, hydro electrical construction monitoring etc.). In order to combine the GNSS observations and high precision measurements using total stations, it is necessary to know the Deflection of the Vertical (DoV), defined as the angle between the gravity vector that coincide with plumb line and the ellipsoidal normal in a given point.

There are different methods of DoV determination using astrogeodetic, gravimetric and GNSS/levelling methods well described by different authors [1-4]. In this paper, the gravimetic and GNSS/leveling methods are compared in order to find a solution for calculation of DoV components for the territory of Republic of Moldova. Also the influences of DoV components on the traditional terrestrial three-dimensional geodetic measurements (azimuth, vertical angles and distance) were analysed using three-dimensional transformations of topocentric geodetic and astronomic (natural) topocentric coordinates taking in account deflections of the vertical and their components.

1. GNSS/Levelling methods of calculations of Deflection of Vertical components

1.1 Calculation of Deflections of Vertical components at Earth's surface

The differential equation of quasigeoid undulation ζ is well known as following [4]:

$$d\zeta = \frac{\partial\zeta}{\partial\phi}d\phi + \frac{\partial\zeta}{\partial\lambda}d\lambda, \tag{1}$$

were is ϕ and λ are the geodetic latitude and longitude, respectively.

The Deflection of Vertical (DoV) components in direction of the meridian ξ and prime vertical η are defined as following [4]:

$$\xi = -\frac{1}{(M+h)}\frac{\partial\zeta}{\partial\phi}, \qquad \eta = -\frac{1}{(N+h)\,cos\phi}\frac{\partial\zeta}{\partial\lambda}, \tag{2}$$

were h is geodetic (ellipsoidal) height, M is radius of curvature of meridian and N is radius of curvature of prim vertical.

Taking in account that ζ could be calculated from GNSS measurements on leveling benchmarks using simple formula:

$$\zeta = h - H, \tag{3}$$

were H is leveled normal height.

The difference of quasigeoid undulation between two points $\Delta\zeta$ from (2-3) is:

$$\Delta\zeta = \zeta_2 - \zeta_1 = h_2 - H_1 - h_2 + H_2 = -(M+h)\Delta\phi\,\xi - (N+h)\,cos\phi\,\Delta\lambda\,\eta, \tag{4}$$

were $\Delta\phi = \phi_2 - \phi_1$ and $\Delta\lambda = \lambda_2 - \lambda_1$.

The parametric observations model could be written as following:

$$\Delta\zeta_i = \mathbf{a}_i\mathbf{x} + \mathbf{v}_i, \tag{5}$$

where $\mathbf{x}$ are unknown parameters ξ and η, $\mathbf{v}_i$ are residuals and $\mathbf{a}_i$ are observation coefficients corresponding to the number of parameters:

$$\Delta\zeta_i = -(M+h)\Delta\phi_i\,\xi - (N+h)cos\phi\,\Delta\lambda_i\,\eta, \tag{6}$$

The parameters are estimated by least squares method:

$$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\Delta\zeta, \tag{7}$$

Introducing the vector of estimated parameters $\mathbf{x}$ into the system of observation equations, is obtained the vector of estimated residuals

$$\mathbf{v} = \mathbf{\Delta\zeta} - \mathbf{Ax}, \tag{8}$$

The estimated residuals **v** are used to calculate the standard deviation:

$$\sigma_0 = \sqrt{\frac{\mathbf{v}^{\mathbf{T}}\mathbf{v}}{n-m}}, \tag{9}$$

where n is the number of GNSS/levelling observations and m is the number of estimated parameters.

For test calculation were selected 5 second order benchmarks of National Levelling Network around control point 7777 (Fig. 1). The GNSS measurements on levelling benchmarks were carried out by INGEOCAD specialists, using GNSS receivers in static mode with 1 hour duration of observations and postprocessed with a connection to EUREF sites and first order National Geodetic Network.

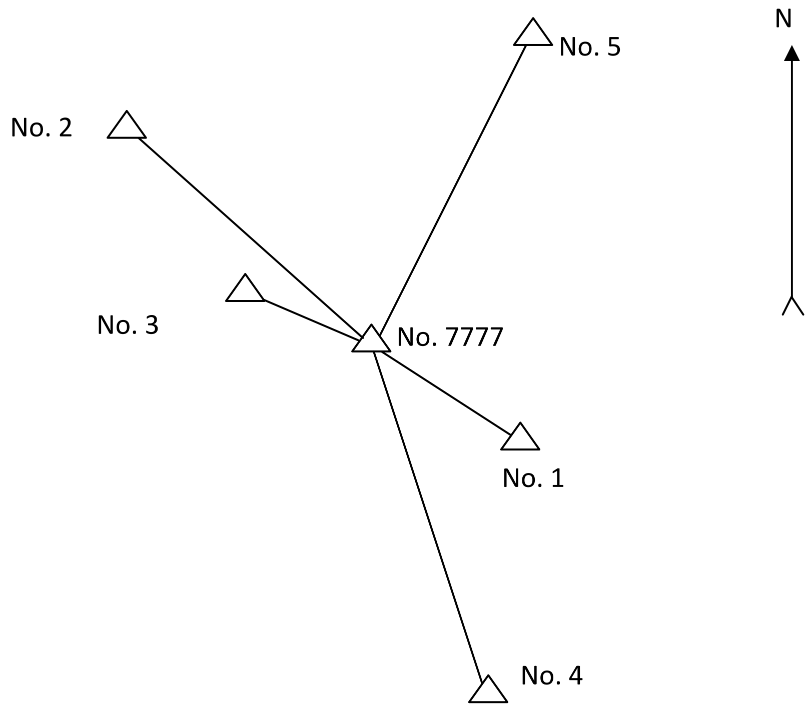

Fig. 1. Levelling benchmarks and GNSS baselines

Table 1. The initial data for DoV components calculation at the Earth's surface

Nr.	ϕ°	λ°	h, m	H, m	ζ, m	Δζ, m	Δϕ″	Δλ″
1	46.829894	29.070224	264.444	233.688	30.756	-0.045	-193.1	266.1
2	47.002924	28.941507	210.184	179.361	30.823	0.022	429.8	-197.3
3	46.918998	28.973992	65.862	35.074	30.788	-0.013	127.6	-80.3
4	46.774183	29.019300	209.510	178.700	30.810	0.009	-393.7	82.8
5	47.022771	29.070997	240.373	209.769	30.604	-0.197	501.2	268.9

Using the method described above, the DoV components on the control point No. 7777 were calculated: $\xi = 1.07'' \pm 0.56''$; $\eta = \pm 3.72'' \pm 0.56''$.

1.2 Calculation of Deflections of Vertical components at quasigeoid's surface

The geometrical relationship between quasigeoid height and deflection of the vertical is defined by the following formulae [5]:

$$v = -\frac{d\zeta}{ds}, \tag{10}$$

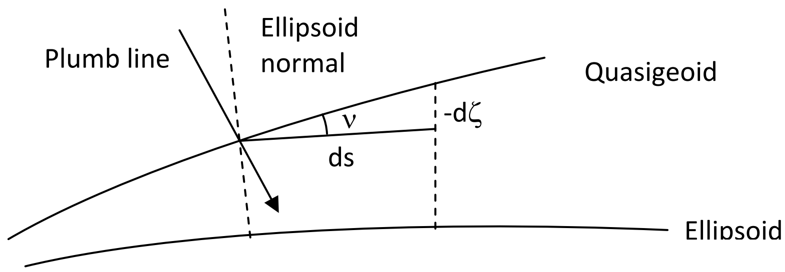

Fig. 2. Geometrical relationship between quasigeoid height and DoV

Deflection of vertical on any geodetic azimuth α direction can be calculated as follows:

$$v = \xi \cos\alpha + \eta \sin\alpha. \tag{11}$$

Taking in account equations (3) and (10) the following formula is obtained:

$$-\frac{\Delta\zeta}{s} = \xi \cos\alpha + \eta \sin\alpha, \tag{12}$$

were s is a distance between two points.

The parametric observations model could be written as following:

$$-\frac{\Delta\zeta_i}{s_i} = \mathbf{a}_i \mathbf{x} + \mathbf{v}_i, \tag{13}$$

where $\mathbf{x}$ are unknown parameters ξ and η, $\mathbf{v}_i$ are residuals and $\mathbf{a}_i$ are observation coefficients corresponding to the number of parameters. The parameters ξ and η, are estimated in planar approximation by least squares method using equations (7-9).

Table 2. The initial data for DoV components calculation at the quasigeoid's surface

Nr.	**N, m**	**E, m**	**h, m**	**H, m**	**ζ, m**	**Δζ, m**	**s, m**	**α°**
1	188243.303	251133.007	264.444	233.688	30.756	-0.045	8205.517	136.151286
2	207401.951	241180.069	210.184	179.361	30.823	0.022	13912.079	342.132268
3	198090.052	243718.787	65.862	35.074	30.788	-0.013	4293.078	336.238756
4	182018.537	247296.687	209.510	178.700	30.810	0.009	12282.190	171.345812
5	209684.414	251008.531	240.373	209.769	30.604	-0.197	16489.175	19.705637

Using the method described above, the DoV components on the control point No. 7777 were calculated: $\xi = 1.18'' \pm 0.66''$; $\eta = 3.00'' \pm 0.66''$.

2. Comparison with gravimetric method of Deflection of Vertical calculations

In order to compare the results of DoV components calculated from GNSS/Levelling the gravimetric deflection of vertical was computed from the Vening-Meinesz integral using classical Molodensky method [5]:

$$\begin{Bmatrix} \xi_{\Delta g} \\ \eta_{\Delta g} \end{Bmatrix} = \frac{\rho''}{4\pi\gamma} \int_{\psi=0}^{\pi} \int_{A=0}^{2\pi} \Delta g \frac{dS(\psi)}{d\psi} \begin{Bmatrix} \cos A \\ \sin A \end{Bmatrix} \sin\psi \, d\psi \, dA, \qquad (14)$$

were $\Delta g = g - \gamma$ are free air gravity anomalies, ψ and A are polar coordinates, and $S(\psi)$ is Stokes function modified by Vening-Meinesz as following,

$$\frac{dS(\psi)}{d\psi} \sin\psi = -\cos^2\frac{\psi}{2}\left\{\operatorname{cosec}\frac{\psi}{2} + 12\sin\frac{\psi}{2} + \frac{3}{1+\sin\frac{\psi}{2}} - 4\sin^2\frac{\psi}{2}\left[8 + 3\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right]\right\}.$$

Taking in account the plan approximation formulae (14) could be simplified for integral calculations in two zones with 5 km and 100 km radiuses (Fig. 3):

$$\begin{Bmatrix} \xi_{\Delta g} \\ \eta_{\Delta g} \end{Bmatrix} = -\frac{\rho''}{4\pi\gamma} \int_{r=0}^{r} \int_{A=0}^{2\pi} \frac{\Delta g}{r} \begin{Bmatrix} \cos A \\ \sin A \end{Bmatrix} dr \, dA. \qquad (15)$$

Calculation of gravimetric DoV was done using 2.5'x2.5'gridded free air gravity anomalies from International Gravimetric Bureau database [7].

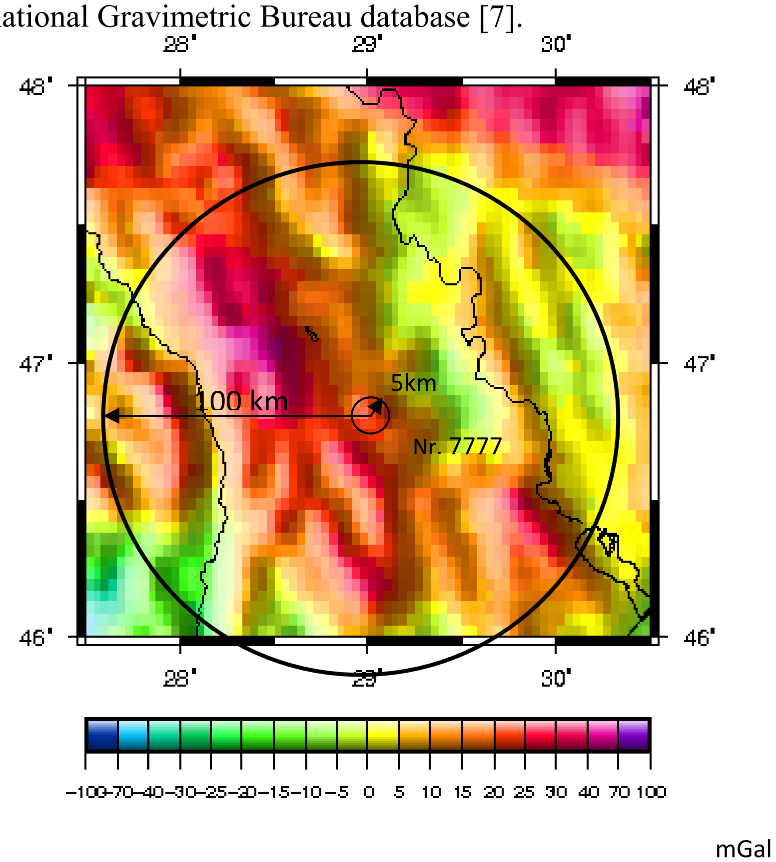

Fig. 3. Calculation of gravimetric DoV from free air gravity anomalies

Using the method described above, the DoV components on the control point No. 7777 were calculated: $\xi_{\Delta g} = 1.13''$; $\eta_{\Delta g} = 2.68''$.

Deflections of the vertical components of the control point were also calculated using Earth Gravitational Model EGM96 (n=m=360) and ELGRAM software from International Service for the Geoid [8] : $\xi_{EGM96} = 0.32''$; $\eta_{EGM96} = 6.34''$.

Taking in account the difference between geometric and gravimetric meridian components of DoV [5]:

$$\Delta\xi = 0.171'' H \sin(2\phi), \tag{16}$$

were H is ellipsoidal height in km, the gravimetric obtained components were corrected. The final calculated deflection components are shown in Table 3.

Table 3. Comparisons of obtained components of DoV using different calculation methods

DoV components	GNSS/ levelling at Earth's surface	GNSS/ levelling at quasigeoid surface	Molodensky gravimetric method	EGM96 n=m=360
ξ	1.07″	1.18″	1.16″	0.35″
η	3.72″	2.99″	2.68″	6.34″

We can notice the remarkable agreement between the first three sets of independently computed vertical deflections special in the meridian component ξ, because of flat surface of quasigeoid in north and south directions from the control point. The substantial differences with vertical deflections computed using EGM96 model could be explained because of low resolution and lack of gravity data of Moldova. The incompliance of prime vertical component η could be explain because the lack of first and second order levelling benchmarks in the east and west part from control point and evident short distances. In conclusion, the most reliable of these four methods is GNSS/Levelling technique. EGM models could be used for the computation of the components of the deflection of vertical combining a geopotential models and free air gravity anomalies (Δg) using well known Remove-Compute-Restore technique.

3. The influence of Deflection of Vertical on traditional geodetic measurements

Some geodetic applications as an underground engineering surveying, hydro electrical construction monitoring, etc. requests to integrate GNSS observations with traditional three-dimensional terrestrial geodetic measurements (horizontal and vertical angles and distance) by using total stations that are related to astronomical (natural) topocentric system of coordinates.

Relationship between geodetic topocentric coordinates *N, E, U,* and astronomical (natural) topocentric system of coordinates N_a , E_a ,U_a are defined by rotation matrix *R* [6]:

$$\begin{bmatrix} N \\ E \\ U \end{bmatrix} = R \begin{bmatrix} N_a \\ E_a \\ U_a \end{bmatrix}; \qquad \begin{bmatrix} N_a \\ E_a \\ U_a \end{bmatrix} = R^T \begin{bmatrix} N \\ E \\ U \end{bmatrix}; \qquad R = \begin{bmatrix} 1 & \nu & \xi \\ -\nu & 1 & \eta \\ -\xi & -\eta & 1 \end{bmatrix}. \tag{17}$$

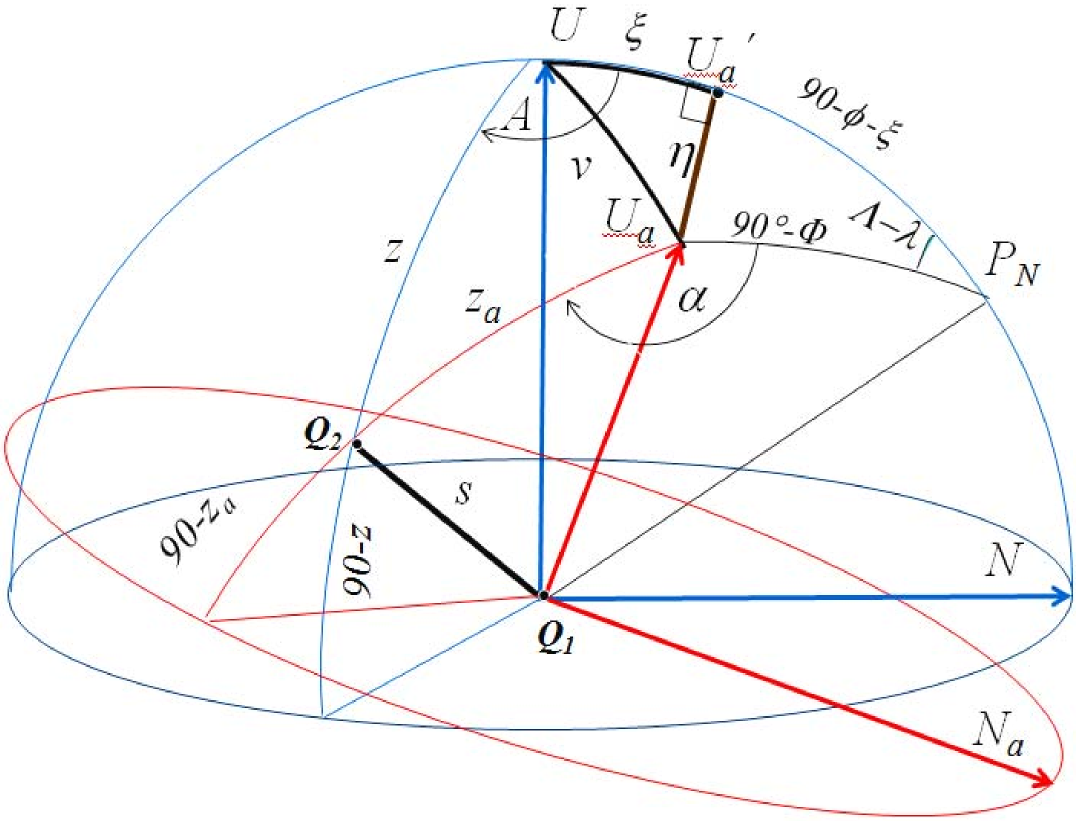

Fig. 4. Astronomical topocentric and geodetic topocentric systems of coordinates

Taking in account that the angle value of the vertical deflection υ is very small, only a few seconds, the spherical triangle UU_aU_a' can be identified with the plane right triangle (Fig. 4), where full deflection of vertical could be calculated as follows applying the Pythagoras rule:

$$\nu = \sqrt{\xi^2 + \eta^2}. \tag{18}$$

Relationship between astronomical azimuth α, vertical angle z_a , distance *s* and natural coordinates N_a , E_a , U_a are well known as the following [6]:

$$\begin{bmatrix} N_a \\ E_a \\ U_a \end{bmatrix} = s\begin{bmatrix} \sin z_a \cos\alpha \\ \sin z_a \sin\alpha \\ \cos z_a \end{bmatrix}; \quad \alpha = arctg\frac{E_a}{N_a}; \quad z_a = arctg\frac{\sqrt{N_a{}^2 + E_a{}^2}}{U_a}; \quad s = \sqrt{N_a{}^2 + E_a{}^2 + U_a{}^2}. \tag{19}$$

Relationship between geodetic azimuth *A*, vertical angle *z* , distance *s* and geodetic topocentric coordinates *N* , *E* , *U* are also well known as the following [6]:

$$\begin{bmatrix} N \\ E \\ U \end{bmatrix} = s\begin{bmatrix} \sin z \cos A \\ \sin z \sin A \\ \cos z \end{bmatrix}; \quad A = arctg\frac{E}{N}; \quad z = arctg\frac{\sqrt{N^2 + E^2}}{U}; \quad s = \sqrt{N^2 + E^2 + U^2}. \tag{20}$$

In order to investigate the influence of vertical deflections and their components on the geodetic azimuth *A*, vertical angle *z* and distance s we simulated measurements of astronomical azimuth α, vertical angle z_a and distance s with differences of deflections of vertical components between two stations $\xi = 0.5''$; $\eta = 0.5''$; $\nu = 0.707''$.

For geodetic vertical angle z = 45°, and distance s = 1000 m differences between geodetic and astronomic azimuth are showed in Fig. 5.

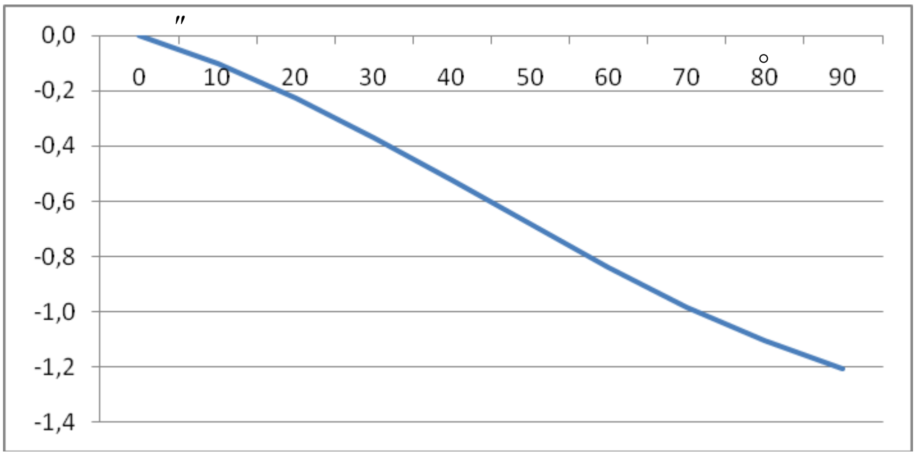

Fig. 5. The influence of deflection of vertical components on geodetic azimuths

For geodetic azimuth A = 45°, and distance s = 1000 m differences between geodetic and astronomic vertical angles are showed in Fig. 6.

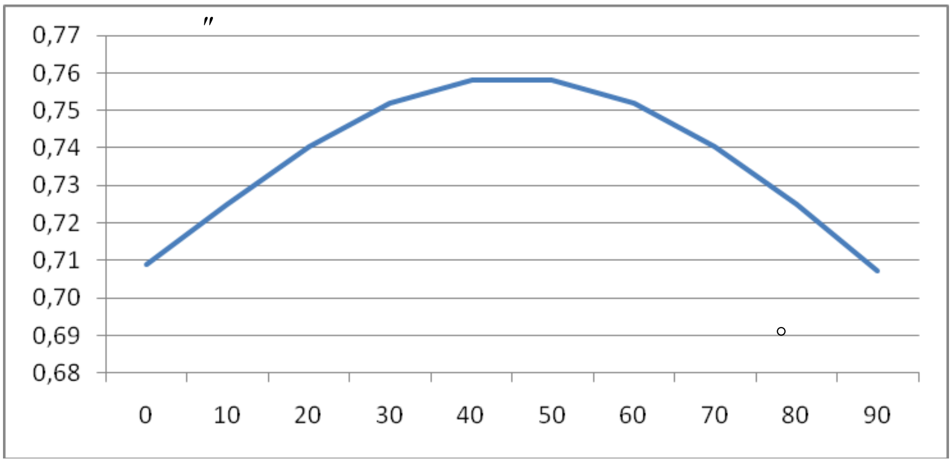

Fig. 6. The influence of deflection of vertical components on geodetic vertical angles

For geodetic azimuth A = 45°, and geodetic vertical angle z = 90° differences in distances are showed in Fig. 7.

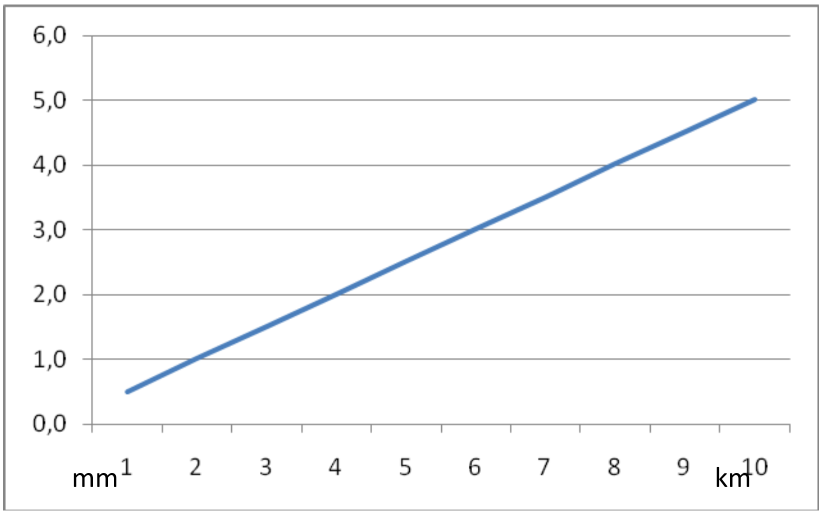

Fig. 7. The influence of deflection of vertical components on distances

4. Conclusions

The agreement between the GNSS/Levelling methods and independent gravimetric technique of computing DoV components shows the necessity to continue investigations in different in regions of the country.

The results of investigations shows the necessity to use I-II order levelling measurements and at least 1 hour GNSS observation in order to determine DoV components with a level of precision comparable to the angular accuracy achievable by 1 arcsec precision total station. The network should be design taking in account the morphology of quasigeoid surface.

In conclusion, the most reliable of these four methods is GNSS/Levelling technique. The preference should be done to the calculation method of Deflections of Vertical components at earth's surface in order to avoid any reduction corrections due to the height above the quasigeoid in case of mountain area.

The simulation examples of the influence of vertical deflections and their components on the geodetic azimuth, vertical angle and distance shows necessity to take in consideration vertical deflections for high precision geodetic applications as an underground engineering surveying, hydro electrical construction monitoring, etc. when GNSS technique is not possible to use directly and integration with traditional three-dimensional terrestrial geodetic measurements is needed.

The future investigations should be oriented to simulate triangles closure in hydro electrical construction micro-triangulation monitoring networks where differences of deflections of vertical are considerable.

1. References

1. Ceylan A.: Determination of the deflection of vertical components via GPS and leveling measurement: A case study of a GPS test network in Konya, Turkey. Scientific Research and Essay Vol.4 (12), December, 2009, pp. 1438-1444.
2. Dumitru P.D., Badescu O., Calin. A. ,:Study regarding the use of the vertical deviation from Romanian historical astro-geodetic network in the geoid model development on the national territory. Romanian Astron. J. , Vol. 26, No. 1, Bucharest, 2016, p. 67–75.

3. *Vittuari L., Tini M. A., Sarti P., Serantoni E., Borghi A., Negusini M. and Guillaume S.: A Comparative Study of the Applied Methods for Estimating Deflection of the Vertical in Terrestrial Geodetic Measurements. Sensors, 2016, 16, 565.*
4. *Soler T., Carlson A. E., Alan G. Evans. Determination of vertical deflections using the Global Positioning System and geodetic levelling. Geophisical research Letters, July, 1989.*
5. *Молоденский М.С., Еремеев В.Ф., Юркина М.И. Методы изучения внешнего гравитационного поля и фигуры Земли // Труды ЦНИИГАиК. - Вып. 131. - М.: Геодезиздат, 1960. - 251 с.*
6. *Машимов М. М. .: Геодезия. Теоретическая геодезия: Справочное пособие/ Под ред. В. П. Савиных и В. Р. Ященко. М. , Недра, 1991, 268 с.*
7. *http://bgi.omp.obs-mip.fr/data-products/Grids-and-models/regional_gravity_anomaly_grids*
8. *http://www.isgeoid.polimi.it/Software/software.html*