GBT-BASED FINITE ELEMENT FORMULATION FOR ELASTIC BUCKLING ANALYSIS OF CONICAL SHELLS

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***Abstract**: The paper presents a finite element formulation based on the Generalised Beam Theory (GBT), for the analysis of elastic buckling behaviour of conical shells under various loading and boundary conditions. The GBT approach provides a general solution for 1st and 2nd order analysis using bar elements capable of describing the global and local deformations. Because of the cross-section variation specific to conical shells, the mechanical and geometric properties are no longer constant along the bar axis as it is the case of cylinders and prismatic thin-walled structures. This formulation is validated by comparison between GBT results and values obtained by means of shell finite element analyses. The study covers simply supported conical shells under axial compression. The study also presents a possible practical application on oil platforms.*

***Keywords:** generalized beam theory, conical shells, finite element method, linear buckling analysis, compression, oil platforms.*

1. Introduction

The stability of cylindrical and conical structures has been studied from the analytical and experimental point of view since the beginning of the 20th century. At first the small deflection theory was used for obtaining bifurcation buckling solutions of shell structures [1, 2]. However, experimental results showed that cylinders buckled at loads well below those predicted by the small deflection theory. Donnell proposed a non-linear theory for circular cylindrical shells under the simplifying shallow-shell hypothesis [3]. Depending on the non-linear strain components considered, other large deflections theories were subsequently proposed by Sanders [4], Flügge [5] and Novozhilov [6]. The Love-Timoshenko non-linear theory [7] is used in the present paper, mainly because, in comparison with the Donnell or Sanders theories, it contains important additional non-linear strain components (Goldfeld summarized in [8] the kinematic relations of these three shell theories).

In this paper the Generalized Beam Theory (GBT) is adapted for the linear stability analysis of truncated conical shells. GBT is an efficient method developed by Richard Schard [9] to analyse the stability of thin-walled prismatic bars, which extends Vlasov's classical beam theory to take into consideration local and distortional cross-section deformation. The first studies which extend GBT to the 1st order analysis and the buckling analysis of cylindrical shells were developed by Christof Schardt and Richard Schardt [9, 10 and 11]. Silvestre developed further this field by studying the buckling (bifurcation) behaviour of circular cylindrical shells subjected to axial compression, bending, compression plus bending and torsion [12]. Also, Silvestre developed a GBT formulation capable of assessing the buckling behaviour of elliptical cylindrical shells subjected to compression [13].

Nedelcu developed a GBT formulation for the buckling analysis of isotropic conical shells under axial compression [14]. As in the GBT formulation for thin-walled prismatic bars with variable cross section developed by the same author [15], the mechanical and geometrical properties are no longer constant along the member's length. However, in the case of conical shells, these properties can be easily defined, because the buckling modes turned out to be a combination of shell-type deformation modes which can be easily pre-determined. In [14], the GBT system of equilibrium equations was solved using the Runge-Kutta Lobatto IIIA collocation method of 4th order [16]. The method proved to have limitations in case of structures subjected to arbitrary loading and boundary conditions. Also, the method proved to be unstable in case of many coupled deformation modes. For these reasons, a GBT-based Finite Element (FE) formulation seems preferable. Such type of FE is already used by many researchers to analyse the buckling behaviour of prismatic thin-walled members and structures under arbitrary loading and boundary conditions [17, 18].

The following paper presents a GBT-based Finite Element (FE) formulation to analyse the elastic buckling behaviour of isotropic conical shells under compression using the Love – Timoshenko large deflection shell theory [5]. The GBT-based FE analysis is implemented in Matlab [19]. For the validation of the results, several models were created in Abaqus [20] using shell finite elements. The methodology was validated by comparing the results obtained using the proposed formulation and SFEA. Finally, the paper presents a possible practical application of the GBT-based FE formulation on the conical pillars of oil platforms.

2. Materials and Methods

2.1. Generalized Beam Theory for conical shells

The GBT adaption for conical shells was presented in detail in [19], therefore this section briefly describes the main aspects. Fig. 1 presents the geometry of a conical member (length *L*, thickness *t*, semi-vertex angle *α*), the global coordinate system x_g, y_g and z_g and the local coordinate system *x*, *θ* and *z*, where: $0 \le x \le L/\cos\alpha$ is the meridional coordinate, $0 \le \theta \le 2\pi$ is the circumferential coordinate and $-t/2 \le z \le +t/2$ is the normal coordinate. The displacements of the structure according to the local coordinate system are as follows: *u* is the displacement along the meridian, *v* is the displacement along the circumference of the cross section and *w* is the displacement along the thickness.

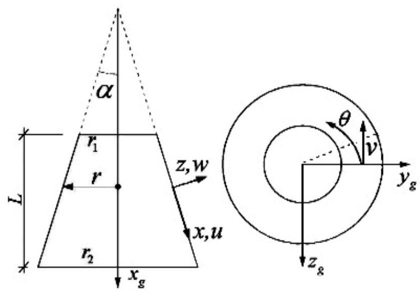

Fig. 1: The geometry of conical shell.

In the GBT adaption for conical shells the strains can be decomposed according to:

$$\{\varepsilon\} = \{\varepsilon^M\} + \{\varepsilon^B\} = \{\varepsilon^M\} + z\{\chi\} \tag{1}$$

where: $\{\varepsilon^M\}$, $\{\varepsilon^B\}$ are the membrane and bending strains, respectively, and $\{\chi\}$ is the vector of variation of curvature with respect to the reference surface. The expressions of the kinematic relationships according to Love – Timoshenko theory [7] are presented in detail in references [14], [22] and [23].

According to GBT, the displacements *u*, *v* and *z* of the middle surface are expressed as a summation of orthogonal functions as follows:

$$u = \sum_{k=1}^{n} \bar{u}_k(x,\theta), \quad v = \sum_{k=1}^{n} \bar{v}_k(x,\theta), \quad w = \sum_{k=1}^{n} \bar{w}_k(x,\theta) \tag{2}$$

where *n* is the number of cross-section deformation modes.

The GBT product formulation is next used to describe each "pure" deformation mode:

$$\begin{aligned} \bar{u}_k(x,\theta) &= u_k(\theta) r(x) \phi_{k,x}(x) \\ \bar{v}_k(x,\theta) &= v_k(\theta) \phi_k(x) \\ \bar{w}_k(x,\theta) &= w_k(\theta) \phi_k(x) \end{aligned} \tag{3}$$

where $u_k(\theta)$, $v_k(\theta)$, $w_k(\theta)$ are the cross-section displacement functions pertaining to mode *k* and $\phi_k(x)$ is the corresponding modal amplitude function defined along the member's length.

2.2. The deformation modes of conical shells

The classical GBT assumes that the linear deformations $\varepsilon_{\theta\theta}^{ML}$ and $\gamma_{x\theta}^{ML}$ respectively are null. In case of conical shells, if the terms *us/r* and *–vs/r* from the kinematic relationships in references [14], [22] and [23] are neglected, then the cross-section displacements v_k and w_k may be expressed with respect the meridional displacement u_k as follows:

$$v_k = -u_{k,\theta} \quad w_k = -\frac{v_{k,\theta}}{c} = \frac{u_{k,\theta\theta}}{c} \tag{4}$$

where $c = \cos\alpha$ and α is the semi-vertex angle of the conical shell.

In case of circular cross sections there are two independent sets of trigonometric functions extensively used in other studies [7, 12] which fulfil the following orthogonality conditions:

$$u_k = \begin{cases} \sin(m\theta), & m = k/2, \quad k = 2,4,6,\dots,n \\ \cos(m\theta), & m = (k-1)/2, \ k = 1,3,5,\dots,n+1 \end{cases} \tag{5}$$

Equation (5) shows that for a given m there are two similar modes with distinct order *k*. The independent trigonometric functions from Eq. (5) define the shell-type deformation modes, who's in-plane configurations are shown in Fig. 2.

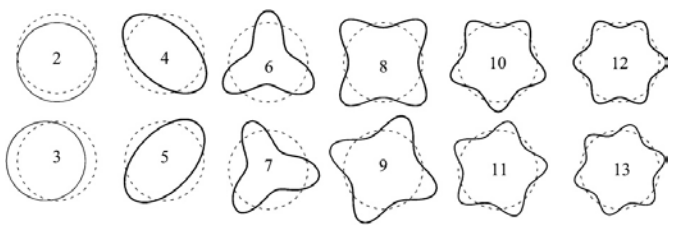

Fig. 2: The shell-type deformation modes.

2.3. The variation of strain energy

Let us consider the first variation of the strain energy according to the linear stability analysis concept:

$$\delta W = \int_L \oint \int_t \left(\sigma_{xx}^L \delta\varepsilon_{xx}^L + \sigma_{\theta\theta}^L \delta\varepsilon_{\theta\theta}^L + \tau_{x\theta}^L \delta\gamma_{x\theta}^L + \sigma_{xx}^0 \delta\varepsilon_{xx}^{NL} + \sigma_{\theta\theta}^0 \delta\varepsilon_{\theta\theta}^{NL} + \tau_{x\theta}^0 \delta\gamma_{x\theta}^{NL} \right) dz\, r\, d\theta\, dx = 0 \tag{6}$$

where σ_{xx}^0, $\sigma_{\theta\theta}^0$ and $\tau_{x\theta}^0$ are the pre-buckling meridional, circumferential and shear stresses, respectively due to the applied external loads.

The constitutive relations are used are presented in detail in reference [23]. Using the kinematic relations from Eq. (1), the constitutive relations from reference [23] and operating all the integrations over the circumference and thickness, the strain energy is expressed as follows:

$$\delta W = \int_L \begin{pmatrix} C_{ik}\phi_{k,xx}\delta\phi_{i,xx} + D_{ik}^1\phi_k\delta\phi_{i,xx} + D_{ik}^2\phi_k\delta\phi_{i,xx} + D_{ki}^2\phi_{k,xx}\delta\phi_i + B_{ik}\phi_k\delta\phi_i \\ +G_{ik}\phi_k\delta\phi_{i,x} + G_{ki}\phi_{k,x}\delta\phi_i + H_{ik}\phi_{k,x}\delta\phi_{i,xx} + H_{ki}\phi_{k,xx}\delta\phi_{i,x} + \\ +X_{jik}^{\sigma x}\phi_{k,x}\delta\phi_{i,x} + X_{jik}^{\sigma\theta}\phi_k\delta\phi_i + X_{jik}^{\tau}\left(\phi_k\delta\phi_{i,x} + \phi_{k,x}\delta\phi_i\right) \end{pmatrix} dx = 0 \tag{7}$$

where C_{ik}, D_{ik}^1, D_{ik}^2, B_{ik}, G_{ik}, H_{ik} are mechanical stiffness matrices related to general warping, twisting and cross sectional distortion, while $X_{jik}^{\sigma x}$, $X_{jik}^{\sigma\theta}$, X_{jik}^{τ} are geometric matrices which take into account the second order effects of the meridional, circumferential and shear stresses, respectively, associated with the deformation mode *j*. The expressions of these matrices are given in detail in references [14], [22] and [23].

2.4. The Finite Element formulation

In this paper a FE formulation previously used for prismatic members ([17, 18]) was adapted for the special case of variable cross-section along the member axis and it is based on the variation form of the strain energy given by Eq. (6). The shape functions to approximate the modal amplitude function $\phi_k(\theta)$ (Eq. (3)) are the classic cubic Hermitian polynomials expressed as follows:

$$\begin{aligned} \psi_1 &= L_e\left(\xi^3 - 2\xi^2 + \xi\right), \quad \psi_2 = 2\xi^3 - 3\xi^2 + 1 \\ \psi_3 &= L_e\left(\xi^3 - \xi^2\right), \qquad \psi_4 = -2\xi^3 + 3\xi^2 \end{aligned} \tag{8}$$

where L_e is the length of the finite element and $\xi = x/L_e$.

Therefore, the modal amplitude function $\phi_k(\theta)$ is approximated in the following form:

$$\phi_k(x) = d_1\psi_1 + d_2\psi_2 + d_3\psi_3 + d_4\psi_4 \tag{9}$$

where $d_1=\phi_{k,x}(0)$, $d_2=\phi_k(0)$, $d_3=\phi_{k,x}(L_e)$ and $d_4=\phi_k(L_e)$ are the degrees of freedom (DOF) of the FE, leading to 4n DOF per mode (2n DOF per node).

By substituting Eq. (9) in Eq. (7) and carrying out the integrations, the finite element matrix bifurcation equation is obtained:

$$\left(\left[K^{(e)}\right]+\lambda\left[G^{(e)}\right]\right)\left\{d^{(e)}\right\}=\{0\} \tag{10}$$

where: $[K^{(e)}]$, $[G^{(e)}]$ are the finite element linear stiffness matrix and geometric stiffness matrix, respectively and $\{d^{(e)}\}$ is the displacement vector.

The GBT-based finite element formulation starts by dividing the member into the desired number of finite elements. The nodal degrees of freedom are identified, and they are grouped in vector $\{d\}$. The finite element stiffness matrices are assembled to form the global linear stiffness matrix $[K]$ and the global geometric stiffness matrix $[G]$. Next the buckling analysis is performed using the pre-buckling stresses and the solution of the eigenvalue problem from Eq. (10) leads to the eigenvalues λ, with the lowest value being the critical value λ_{cr}, and the corresponding buckling modes represented by the eigenvectors $\{d\}$. To determine a deformed configuration of the conical shell, the modal amplitude function $\phi_k(x)$ is found from the superposition of the shape functions $\psi_i(\xi)$ using Eq. (9). Next, the displacement field is found using Eq. (3) and Eq. (2).

3. Results and Discussion

3.1. Numerical examples

For the numerical examples presented in this paragraph let us consider the conical shell from Fig. 3. The analysed steel conical shell ($E=210\ GPa$, $\nu=0,3$) has the wall thickness $t=1\ mm$ and the length $L=1200\ mm$. The starting radius is taken $r_1=50\ mm$, while the end radius r_2 is variable. The pre-buckling meridional stress is calculated as following:

$$\sigma_{xx}^{0}=P_0\big/\left(2\pi\cdot r\cdot t\cdot\cos\alpha\right) \tag{11}$$

where $P_0=1\ kN$ and α is the semi-vertex angle.

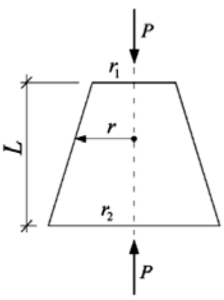

Fig. 3: Conical shell under axial compression.

In the buckling analysis of the conical shell described above, finite elements with 2 nodal degrees of freedom were used, which means that each finite element has 4 degrees of freedom. Fig. 4 shows the critical buckling modes of the simply supported conical shells for different values of the bottom radius r_2 and the corresponding critical buckling coefficients λ_c resulted from SFEA. The same cases were studied in reference [14] using numerical integration instead of FEM and there are no significant differences between the results as seen in Table 1. In Fig. 5 are the normalized graphs of the modal amplitude function $\phi_k(x)$.

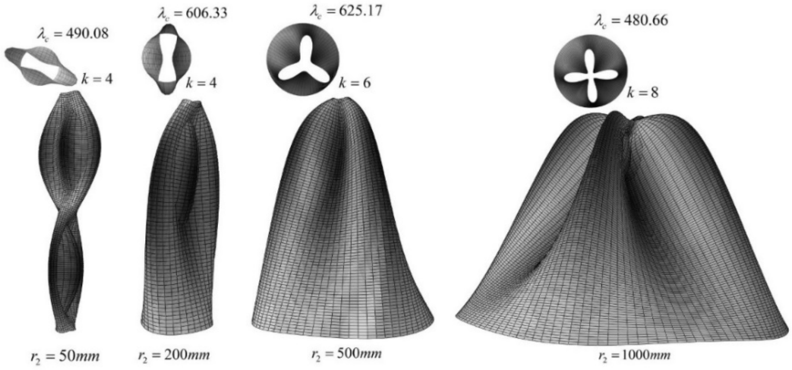

Fig. 4: The critical buckling modes of simply supported conical shells resulting from SFEA [20].

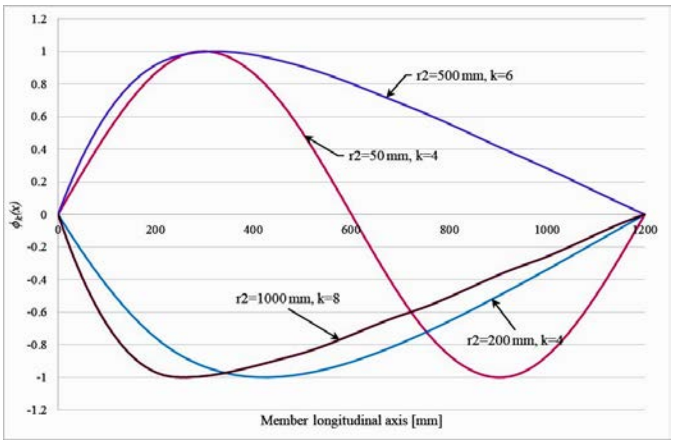

Fig. 5: Simply supported conical shells: the graphs of the modal amplitude functions $\phi_k(x)$ resulted from Matlab [19].

Table 1: SFEA vs GBT-FEM results for long simply supported conical shells.

r_2[mm]	λ_c SFEA	λ_c GBT-FEM	Differences (2) vs. (3)	λ_c GBT-Runge Kutta	Differences (2) vs. (5)	k	n_{hw}
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*
50	490.40	492.54	0.43%	491.35	0.19%	4	2
200	606.33	629.77	3.72%	625.18	3.11%	4	1
500	627.06	640.83	2.15%	633.97	1.10%	6	1
1000	480.97	471.42	2.03%	466.53	3.00%	8	1

Table 1 presents the critical buckling coefficients resulted from SFEA, the ones resulted from the GBT-based FE formulation and, respectively from the Runge-Kutta numerical method [14]. The table also shows the differences between the results obtained from the proposed GBT-based FE formulation and the ones obtained from SFEA. According to the Table 1, both GBT-based formulations provide similar results, but the proposed GBT-based FE formulation is superior in terms of convergence speed, of versatility of load and boundary condition cases and, the most important, of the possibility of coupled instabilities analysis. Table 1 also shows the order of the cross-section deformation mode k and the corresponding number of longitudinal half-waves n_{hw}. The values from the table were determined for 5 FE in the GBT-based FE formulation resulting 6 nodes and 12 DOFs for all the cases. In SFEA, for the cylindrical shell 22,506 DOFs were used. The number of DOFs used in SFEA reached 52,974 for the conical shell with the bottom radius r_2=*1000 mm*.

3.2. Possible practical applications

A possible practical application of the GBT-based FE formulation for conical shells would be for the structural analysis of the pillars belonging to oil platforms. Oil platforms are megastructures designed not only to support the machines used to extract the fossil fuels found in the depths of the seas and oceans, but also to withstand the hostile marine environment. However, these structures must be safe from explosions, fire hazards, loss of control over the oil rig or oil spills. The safety regulations of oil platforms available in the European Union are found in the European Commission's Directive 2013/30/EU [24]. Fig. 6 presents the Romanian oil platforms Doina and Ana built in March 2021 and located in the Midia offshore perimeter of the Black Sea at 120 km away from the coast line [25]. Given the strict safety regulations, the stability of the pillars of oil platforms is very important. This is where the GBT-based FE formulation could provide insights on the deformed configuration of the conical pillars. But since the loading conditions given by the hydrostatic pressure of the sea water combined with wind loads are complicated, the GBT-based FE formulation would require an adaption for the horizontal loads.

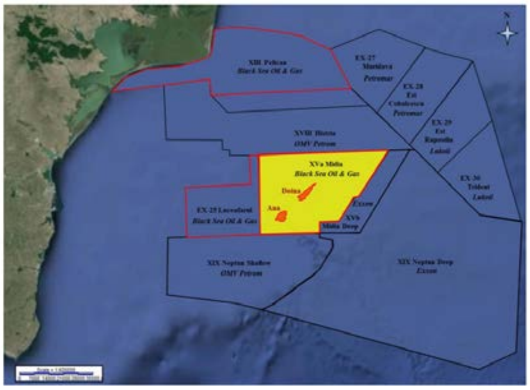

Fig. 6: The Romanian oil platforms Doina and Ana in the Midia offshore perimeter [25].

4. Conclusions

GBT provides a very efficient mean to calculate bifurcation loads of thin-walled bars. Initially it was believed that GBT could only be applied to members having constant cross section along the longitudinal axis, but recent studies proved that it can be extended to structural members with variable cross-section [14, 15]. This paper presented a GBT-based FE formulation to analyse the buckling behaviour of isotropic conical shells under compression. The advantages of using this formulation in comparison with the classic SFEA are the following: much fewer DOFs are necessary to obtain similarly accurate results and in a coupled instability problem it provides the degree of modal participation which offers important information about the buckling behaviour.

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