THE INFLUENCE OF THE PROJECTION PLANE ALTITUDE ON THE DISTORTIONS IN OBLIQUE DOUBLE CONICAL MAP PROJECTIONS

Doina VASILCA, Dr. Eng. Lecturer, Technical University of Civil Engineering, Faculty of Geodesy, Romania, doina.vasilca@utcb.ro

Abstract*: In double map projections, the Earth's surface – approximated as a revolution ellipsoid – is first represented on a sphere that is then subsequently represented on the plane of the map projection. In the present study, the ellipsoid is conformally represented on a sphere, which introduces linear and area distortions. Herein we have studied the representation of the sphere on the lateral surface of a cone chosen such that the tangent parallel passes through the middle of the area of interest. Additionally, in this case, linear and area distortions also occur. The total distortions produced by double map projections must include both the distortions arising from representing the ellipsoid on the sphere, as well as those appearing when representing the sphere on the plane. Said distortions depend on the position of the point on the surface of the ellipsoid or the sphere, as well as on the latitude of the parallel which is represented as undistorted.*

The projection plane altitude also has a influence on the produced distortions, as can be demonstrated by changing the altitude of the projection plane. In the current article, we present this scenario for several road sections in Romania that were represented in local projection plans located at the average altitude of their respective location.

Keywords: *area distortions; conic map projection; double map projections; linear distortions; projection plane altitude*

1. Introduction

The choice of projection for making a map is made according to the geographical position of the territory to be represented, its extent and shape, considering that the distortions produced during its representation in the projection plane should be as small as possible. For territories located at medium latitudes, as is the case of Romania, oblique projections are preferred. In the context of these map projections, the Earth – approximated to a revolution ellipsoid – is first represented on a sphere, after which the sphere is represented on the plane thereby resulting in a double map projection.

In the present work, we analysed double oblique conformal map projections. Thus, the surface of the ellipsoid was conformally represented on a sphere, following which the sphere was represented in an oblique conformal conic map projection. Consequently, it can be said that the double projection is a conformal one [1].

The total distortions introduced during the representation on the plane must comprise of the distortions produced when representing the ellipsoid on the sphere added to those produced when representing the sphere on the plane.

In the case study outlined herein, we analysed the distortion values for several altitudes of the map projection plane, which proves useful for constructions extending over large areas or lengths situated at high altitudes.

2. Materials and Methods

The conformal representation of the ellipsoid on the sphere is made under the condition that a parallel of latitude φ_k is represented undistorted. This means that, with the exception of all angles and the length of the mentioned parallel, the distances in any direction, as well as areas will be represented as distorted. In order to reduce these distortions, a local map projection can be used, hence why we proposed that the latitude of the undistorted parallel be equal to the average latitude of the analysed area.

Regarding the oblique conformal conic map projection, we also proposed a local map projection with the tangent circle, almucantar, crossing through the middle of the region to be mapped.

The distortions produced when conformally representing the ellipsoid on the surface of a sphere are calculated with the following formulas [2]:

$$m = n = \frac{R}{a}\left(1 + \frac{e^2}{2}\sin^2\varphi\right), \tag{1}$$
$$p = m^2 \tag{2}$$

where: m is the linear scale factor along the meridians;
n is the linear scale factor along the parallels;
p is the area scale factor;
a – semimajor axis of the ellipsoid;
e – first eccentricity of the ellipsoid;
φ – the latitude of the point where distortions are calculated.
R is the radius of the sphere and is calculated as follows:

$$R = a/\left(1 + \frac{e^2}{2}\sin^2\varphi_k\right) \tag{3}$$

In the oblique conformal conic map projection, distortions are determined via the following relations [3]:

$$\mu_1 = \mu_2 = \frac{\alpha k}{R\sin z}\tan^{\alpha}\left(\frac{z}{2}\right) \tag{4}$$
$$p = \mu_1^2 = \mu_2^2 \tag{5}$$
$$\alpha = \text{const.} = \cos z_k \tag{6}$$
$$k = const. = R\tan z_k ctan^{\cos z_k}\left(\frac{z_k}{2}\right) \tag{7}$$

where: μ_1 is the linear scale factor along verticals;
μ_2 is the linear scale factor along almucantars;
z is the zenith angular distance;
z_k is the angular zenith distance of the almucantar represented undistorted.

Since the double conical projection is conformal, the linear distortions at any point have the same value in any direction radiating from that point. Thus, the total linear distortion, μ_t, is calculated as the product between the linear distortion produced when representing the ellipsoid on the sphere and the linear distortion produced when representing the sphere on the plane.

$$\mu_t = m \cdot \mu_1 \tag{8}$$

Consequently, the total area distortion, p_t, is calculated using the formula:

$$p_t = \mu_t^2 \tag{9}$$

To highlight the influence of the average altitude, h, above the ellipsoid/sphere, of the areas where these distortions are studied, the value of this altitude must be added to the semi-major, a, and semi-minor, b, axes of the ellipsoid according to which the radius of the sphere, R, is then calculated [4].

In addition, when applying the plotting elements in the field, some corrections must be made, namely the distances on the surface of the ellipsoid/sphere, d_3, must be brought to the ground level, d_1.

The relation between the ellipsoid/sphere arc length, d_3, and the sloped distance at the ground level, d_1, is given by the following equations [5]:

$$d_2 = \sqrt{\frac{d_1^2-(H_2-H_1)^2}{[1+(H_1/R)][1+(H_2/R)]}} \tag{10}$$

$$d_3 = d_2 + \frac{d_2^3}{24R^2}, \tag{11}$$

where: d_1 is the sloped distance at the ground level;
d_2 represents the ellipsoidal/spherical chord;
d_3 is the ellipsoid/sphere arc length;
H_1, H_2 are the ellipsoid hights at the end points of the line;
R is the radius of curvature of the ellipsoid along the line.

To exemplify the influence of the altitude at which the area to be represented is located on the distortions produced at the representation in the map projection plane, three road sections in Romania located at high altitudes were analysed. Furthermore, for a section denominated P-F-R between the following points P: $\varphi = 44.90^o, \lambda = 26.00^o$, F: $\varphi = 45.7008^o, \lambda = 27.1683^o$ and R: $\varphi = 46.9333^{o,}, \lambda = 27.1694^o$ we calculated the linear and area distortions, while assuming an average altitude of h = 0 m, h = 500 m, h = 1000 m and h = 1500 m.

Additionally, for three road sections in our country, Borșa – Prislop Pass – Iacobeni, Rucar – Bran Pass – Bran, Novaci – Oașa Dam – Sebeș, which are located at high altitudes, the relative linear and area distortions were calculated at the average altitude of each section.

Tables 6, 7 and 8 contain the values obtained for the zenith distances of the almucantars between which these sections strictly fall, namely: Borsa – Prislop Pass – Iacobeni: [z_k-3', z_k+1'], Rucar – Bran: [z_k-1', z_k+1'] and Novaci – Oasa Dam – Sebeș: [z_k-2', z_k+1'].

The differences between the distances reduced to the sphere and the inclined lengths on the actual land surface were calculated for the above-mentioned road section, as well as for the Ploiesti – Focsani – Roman section and these are contained in Table 9.

3. Results and Discussion

For the P-F-R section, the angular zenith distance of the almucantar represented undistorted was z_k=1.517434°. The total linear and area distortions were calculated for a strip extended to z_k±5' of the undistorted almucantar (which corresponds to approximately ±9 km) to cover the possible windings of the road from z_k. Our results are presented in Table 1 (altitude h=0 m), Table 2 (altitude h=500 m), Table 3 (altitude h=1000 m) and Table 4 (altitude h=1500 m).

Table 1. Total relative linear distortions D[cm/km] and total relative area distortions P[m^2/km^2]
z_k=1.517434°, h=0 m

	Point	φ [°]	φ' [°]	z [°]	μ total	D total [m/km]	P total	P total [m^2/km^2]
z_k-5'	P	44.9771	44.7848	1.4341	0.999950	-4.95	0.999901	-99.01
	F	45.7347	45.5424	1.4341	0.999995	-0.53	0.999989	-10.67
	R	46.8995	46.7076	1.4341	1.000063	6.25	1.000125	125.02
z_k	P	44.9000	44.7077	1.5174	0.999945	-5.51	0.999890	-110.16
	F	45.7008	45.5086	1.5174	0.999992	-0.84	0.999983	-16.77
	R	46.9333	46.7414	1.5174	1.000063	6.34	1.000127	126.80
z_k+5'	P	44.8229	44.6306	1.6008	0.999941	-5.85	0.999883	-117.07
	F	45.6669	45.4746	1.6008	0.999991	-0.93	0.999981	-18.65
	R	46.9671	46.7752	1.6008	1.000066	6.64	1.000133	132.81

Table 2. Total relative linear distortions D[cm/km] and total relative area distortions P[m^2/km^2]
z_k=1.517434°, h=500 m

	Point	φ [°]	φ' [°]	z [°]	μ total	D total [m/km]	P total	P total [m^2/km^2]
z_k-5'	P	44.9771	44.7848	1.4341	0.999950	-4.95	0.999901	-99.00
	F	45.7347	45.5424	1.4341	0.999995	-0.53	0.999989	-10.67
	R	46.8995	46.7076	1.4341	1.000063	6.25	1.000125	125.01
z_k	P	44.9000	44.7077	1.5174	0.999945	-5.51	0.999890	-110.15
	F	45.7008	45.5086	1.5174	0.999992	-0.84	0.999983	-16.77
	R	46.9333	46.7414	1.5174	1.000063	6.34	1.000127	126.79
z_k+5'	P	44.8229	44.6306	1.6008	0.999941	-5.85	0.999883	-117.06
	F	45.6669	45.4746	1.6008	0.999991	-0.93	0.999981	-18.65
	R	46.9671	46.7752	1.6008	1.000066	6.64	1.000133	132.80

Table 3. Total relative linear distortions D[cm/km] and total relative area distortions P[m^2/km^2]
z_k=1.517434°, h=1000 m

	Point	φ [°]	φ' [°]	z [°]	μ total	D total [m/km]	P total	P total [m^2/km^2]
z_k-5'	P	44.9771	44.7848	1.4341	0.999951	-4.95	0.999901	-98.99
	F	45.7347	45.5424	1.4341	0.999995	-0.53	0.999989	-10.67
	R	46.8995	46.7076	1.4341	1.000063	6.25	1.000125	125.01
z_k	P	44.9000	44.7077	1.5174	0.999945	-5.51	0.999890	-110.14
	F	45.7008	45.5086	1.5174	0.999992	-0.84	0.999983	-16.77
	R	46.9333	46.7414	1.5174	1.000063	6.34	1.000127	126.78
z_k+5'	P	44.8229	44.6306	1.6008	0.999941	-5.85	0.999883	-117.05
	F	45.6669	45.4746	1.6008	0.999991	-0.93	0.999981	-18.65
	R	46.9671	46.7752	1.6008	1.000066	6.64	1.000133	132.79

Table 4. Total relative linear distortions D[cm/km] and total relative area distortions P[m^2/km^2]
z_k=1.517434°, h=1500 m

	Point	φ [◦]	φ' [◦]	z [°]	μ total	D total [m/km]	P total	P total [m^2/km^2]
z_k-5'	P	44.9771	44.7848	1.4341	0.999951	-4.95	0.999901	-98.98
	F	45.7347	45.5424	1.4341	0.999995	-0.53	0.999989	-10.67
	R	46.8995	46.7076	1.4341	1.000062	6.25	1.000125	125.00
z_k	P	44.9000	44.7077	1.5174	0.999945	-5.51	0.999890	-110.13
	F	45.7008	45.5086	1.5174	0.999992	-0.84	0.999983	-16.77
	R	46.9333	46.7414	1.5174	1.000063	6.34	1.000127	126.77
z_k+5'	P	44.8229	44.6306	1.6008	0.999941	-5.85	0.999883	-117.04
	F	45.6669	45.4746	1.6008	0.999991	-0.93	0.999981	-18.64
	R	46.9671	46.7752	1.6008	1.000066	6.64	1.000133	132.78

Table 5. The influence of altitude on total relative linear distortions D[cm/km] and total relative area distortions P[m^2/km^2] at points that define P-F-R section, z_k=1.517434°

					Differences between h=500m and h=0m		Differences between h=1000m and h=0m		Differences between h=1500m and h=0m	
	Point	φ [◦]	φ' [◦]	z [°]	Δ D [cm/km]	Δ P [m^2/km^2]	Δ D [cm/km]	Δ P [m^2/km^2]	Δ D [cm/km]	Δ P [m^2/km^2]
z_k-5'	P	44.9771	44.7848	1.4341	0.00	0.01	0.00	0.02	0.00	0.03
	F	45.7347	45.5424	1.4341	0.00	0.00	0.00	0.01	0.00	0.01
	R	46.8995	46.7076	1.4341	0.00	-0.01	0.00	-0.02	0.00	-0.02
z_k	P	44.9000	44.7077	1.5174	0.00	-0.02	0.00	-0.01	0.00	0.00
	F	45.7008	45.5086	1.5174	0.00	0.00	0.00	0.00	0.00	0.00
	R	46.9333	46.7414	1.5174	0.00	0.02	0.00	0.01	0.00	0.00
z_k+5'	P	44.8229	44.6306	1.6008	0.00	0.01	0.00	0.02	0.00	0.03
	F	45.6669	45.4746	1.6008	0.00	0.00	0.00	0.01	0.00	0.01
	R	46.9671	46.7752	1.6008	0.00	-0.01	0.00	-0.02	0.00	-0.03

As can be seen from Table 5, if the semi-axes of the ellipsoid are modified by adding the value of mean altitude, then the influence of altitude on the total linear distortion is 0.00 cm/km, and the overall area distortions vary between -0.03 m^2/km^2 and +0.03 m^2/km^2.

Table 6. The influence of altitude on the total relative linear and relative area distortions at the points that define the Borșa – Prislop Pass – Iacobeni section

Point	z	z[°]	Altitude 0 m		Altitude 1000 m		Δ D [cm/km]	Δ P [m^2/km^2]
			D tot [cm/km]	P tot [sqm/sqkm]	D tot [cm/km]	P tot [m^2/km^2]		
Borsa	zk-3'	1.0135	0.31	6.13	0.31	6.13	0.00	0.01
	zk+1'	1.1302	0.65	12.91	0.65	12.91	0.00	0.00
Prislop Pass	zk-3'	1.0135	0.06	1.20	0.06	1.20	0.00	0.00
	zk+1'	1.1302	0.38	7.66	0.38	7.66	0.00	0.00
Iacobeni	zk-3'	1.0135	-1.02	-20.32	-1.02	-20.32	0.00	0.00
	zk+1'	1.1302	-0.76	-15.23	-0.76	-15.22	0.00	0.01

Table 7. The influence of altitude on the total relative linear and relative area distortions at the points that define the Rucăr – Bran Pass – Bran

Point	z	z[°]	Altitude 0 m		Altitude 900 m		Δ D [cm/km]	Δ P [m²/km²]
			D tot [cm/km]	P tot [m^2/km^2]	D tot [cm/km]	P tot [m^2/km^2]		
Rucăr	zk-1'	0.996725	-0.37	-7.31	-0.37	-7.31	0.00	0.01
	zk+1'	1.030058	-0.22	-4.36	-0.22	-4.35	0.00	0.00
Bran Pass	zk-1'	0.996725	-0.07	-1.38	-0.07	-1.38	0.00	0.00
	zk+1'	1.030058	0.09	1.78	0.09	1.78	0.00	0.00
Bran	zk-1'	0.996725	0.21	4.22	0.21	4.22	0.00	0.00
	zk+1'	1.030058	0.38	7.57	0.38	7.57	0.00	0.00

Table 8. The influence of altitude on the total relative linear and relative area distortions at the points that define the Novaci – Oașa – Sebeș section

Point	z	z[°]	Altitude 0 m		Altitude 700 m		Δ D [cm/km]	Δ P [m²/km²]
			D tot [cm/km]	P tot [m^2/km^2]	D tot [cm/km]	P tot [m^2/km^2]		
Novaci	zk-2'	10.8257	-2.30	-46.05	-2.30	-46.04	0.00	0.01
	zk+1'	10.8757	-2.30	-45.99	-2.30	-45.98	0.00	0.01
Oașa Dam	zk-2'	10.82566	0.07	1.38	0.07	1.38	0.00	0.00
	zk+1'	10.87566	0.08	1.66	0.08	1.66	0.00	0.00
Sebeș	zk-2'	10.82566	2.23	44.62	2.23	44.61	0.00	0.00
	zk+1'	10.87566	2.25	45.10	2.25	45.09	0.00	0.00

For the cases analysed in Tables 6, 7 and 8, modifying the dimension of the ellipsoid with the value of the average altitude of the road section influences only the relative area distortion up to +0.01m^2/km^2.

Table 9. Differences, Δ D, between the sphere arc length, d_3, and the sloped distance at the ground level, d_1

Borsa - Prislop Pass - Iacobeni road section, R=6366640.260 m							
Point	H [m]	d_1 [m]	d3 [m]	Δ D [m]	Δ H [m]	D [cm/km]	Slope [%]
Borsa	673	15400	15382.069	-17.931	743.000	1.16	4.825
Prislop Pass	1416	39900	39895.834	-4.166	-581.000	0.10	-1.456
Iacobeni	835						
Novaci - Oasa Dam - Sebes road section, R=6367375.902 m							
Novaci	650	45900	45896.112	-3.888	605.000	0.09	1.318
Oasa Dam	1255	41600	41588.221	-11.779	-993.000	0.28	-2.387
Sebes	262						
Rucar - Bran Pass - Bran road section, R=6367427.035 m							
Rucar	687	12100	12089.122	-10.878	513.000	0.90	4.240
Bran Pass	1200	9350	9339.783	-10.217	-437.000	1.09	-4.674
Bran	763						
Ploiesti – Focsani - Roman road section, R=6367276.161 m							
Ploiesti	165	144000	144003.020	3.020	-119.000	0.02	-0.083
Focsani	46	130000	130002.173	2.173	149.000	0.02	0.115
Roman	195						

The distances in the projection plane must be brought to the actual ground surface when they are being set out. From Table 9, the influence the terrain slope elicits on these distances can be seen for the analysed road sections. If the difference between the arc sphere length and the slope distance at the ground level is expressed as a linear distortion, then D[cm/km] can be observed to vary between 0.02 cm/km for a 0.01% slope up to 1.16 cm/km for a 4.8% slope.

4. Conclusions

When setting out constructions that span large lengths or surfaces, one must consider both the distortions introduced by the map projection and the fact that distances must be set out according to the real surface of the land while also considering its slope.

The distortions produced during the representation in the projection plane are only influenced by the position of the points on the surface of the ellipsoid if the semi-major axes of the ellipsoid are modified by the altitude of the points where these distortions are calculated. Therefore, in the context of the section defined by the points P ($\varphi = 44.90^o, \lambda = 26.00^o$), F ($\varphi = 45.7008^o, \lambda = 27.1683^o$) and R ($\varphi = 46.9333^{o,}, \lambda = 27.1694^o$), having the angular zenithal distance of the tangent almucantar $z_k = 1.517434^o$, if the altitude is considered 0 m, then the linear relative distortions will be $D \in [-5.85\, cm/km\,, +6.64\, cm/km]$, while the relative area ones will be $P \in [-117.07\, sqm/sqkm, +132.80\, sqm/sqkm]$. The relative area distortions vary only by ±0.03 m^2/km^2 at the mean altitude of 1500 m, when the semi-axes of the ellipsoid are increased by this value.

The influence of altitudes on relative linear and total relative area distortions was also studied for three road sections, of different lengths and located at high altitudes. These are:

Borsa – Prislop Pass– Iacobeni, located at an average altitude of 1000 m and the length of the arc that passes through these three points being 55 km;

Bran – Rucar, located at an average altitude of 900 m, with the length of the circular arc being 21 km;

Novaci – Oasa Dam– Sebes, with an average altitude of 700 m and the length of the circular arc being 87 km.

For the first section, Borsa – Prislop Pass – Iacobeni, the distortions were calculated in a band that stretches from z_k-3' to z_k+1' so as to cover all the sinuosity of the communication path. The relative linear distortions varied from -1.02 cm/km to +0.65 cm/km at 0 m altitude. The area distortions varied between -20.32 sqm/sqkm and +12.91 sqm/sqkm at 0 m altitude.

For the Rucar – Bran section, the linear relative distortions at 0 m altitude varied from -0.37 cm/km to +0.38 cm/km, for the band z_k-1', z_k+1' on which the respective route falls. For the same band, the relative area distortions at 0 m altitude showed values between -7.31 sqm/sqkm and +7.57 sqm/sqkm.

For the Novaci – Oasa Dam – Sebes section, the values of the relative linear distortions, calculated in the band z_k-2', z_k+1', varied between -2.30 cm/km and +2.25 cm/km at the altitude of 0 m. The relative area distortions had values between -46.05 sqm/sqkm and +45.10 sqm/sqkm at the altitude of 0 m.

In all three cases mentioned above, the semi-axes of the ellipsoid have been increased with the average altitude of each section, and consequently the sphere on whose surface it is represented was modified as well. Therefore, the relative linear distortions have not changed compared to the case when the altitude was 0 m. A change was observed only in terms of area distortions, namely 0.01 sqm/sqkm.

Regarding the influence of the terrain slope on the differences between the distances reduced at the ellipsoid level and the inclined lengths on the actual terrain surface varied depending on the slope of the land.

Considering all of the above, taking into account the average altitude (by adding it to the semi-major axes of the ellipsoid) at which the plotting elements of large construction projects spanning large distances/areas are calculated contributes not only to the correct plotting of the respective objectives on the ground, but implicitly also reduces construction costs as well.

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6. References

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