DUAL ALGEBRA REPRESENTATIONS FOR VOLUMES DETERMINATION. AN OVERVIEW

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***Abstract**: The volume, traditionally, as a mathematical model, is a geometric property encoded using linear algebra through determinants and matrix operations. This paper presents a method to calculate volumes directly using dual algebra, bypassing coordinate-dependent discretization and leveraging its inherent capacity to represent surfaces, intersections, and enclosed spaces through concise, error-aware computations. The purpose of this article is to demonstrate the application of dual algebra as a unified computational framework for volume determination, highlighting its mathematical advantages over traditional methods reliant on localized coordinate systems and iterative geometric approximations. Dual algebra, which inherently integrates geometric transformations and error propagation within its algebraic structure, offers a streamlined approach to modelling complex surfaces and calculating enclosed volumes without the need for intermediate coordinate transformations or ad hoc adjustments. Dual algebra operates directly on geometric primitives (e.g., planes, points, and vectors) using a coordinate-free formalism. This enables precise, closed-form solutions for volume calculations, even in irregular geometries. Dual algebra emerges as a mathematically alternative for volume determination, with broad applicability in geomatics, engineering, and environmental modelling.*

***Keywords:** dual algebra; volume determination*

1. Introduction

Volume computation is a fundamental problem in mathematics and physics, appearing in fields such as geometry, linear algebra, and 3D cadastres, which is a significant focus in land administration [1]. Traditionally, the volume of an-dimensional parallelepiped spanned by vectors is obtained via determinants. However, this method depends on a chosen coordinate system. Dual algebra, by contrast, provides an invariant approach using wedge products of dual vectors, enabling calculations independent of coordinates.

The paper is structured as follows: in the second section, a mathematical preliminary for dual algebra (dual numbers, dual vectors, dual tensors) is introduced. Using these mathematical results, we investigate in the third sectionthe volume written with properties of dual algebra. Using these findings, we investigate in the fourth section the comparison of classical volume computation using determinants and volume computation using dual vectors.

2. Mathematical preliminaries

The following section outlines the properties of dual numbers and dual vectors with details in references [2], [3], [4] and [5].

Dual numbers

We denote the set of real dual numbers as

$$\underline{\mathbb{R}} = \mathbb{R} + \varepsilon\mathbb{R} = \{\underline{a} = a + \varepsilon a_0 \mid a, a_0 \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\} \tag{1}$$

where $a = \mathrm{Re}(\underline{a})$represents the real part of $\underline{a}$ and $a_0 = Du(\underline{a})$ the dual part. The operations of addition and multiplication on $\underline{\mathbb{R}}$generate a ring with zero divisors structure for $\mathbb{R}$. In this paper, we primarily utilize the magnitude and inverse properties. The magnitude of a dual number given by $|\underline{a}| = |a| + \varepsilon \mathrm{sgn}(a) a_0$satisfies$|\underline{a}|^2 = \underline{a}^2$. The invers of $\underline{a}$, denoted by $\underline{a}^{-1} \in \overline{\mathbb{R}}$, exists if and only if $\mathrm{Re}(\underline{a}) \neq 0$ and is calculatedas$\underline{a}^{-1} = \frac{1}{\underline{a}} = \frac{1}{a} - \varepsilon\frac{a_0}{a^2}$. A dual number$\underline{a} \in \underline{\mathbb{R}}$ is a zero divisor if and only if $\mathrm{Re}(\underline{a}) = 0$. Consequently,$(\underline{\mathbb{R}}, +, \cdot)$ is a commutative and unitary ring and any element $\underline{a} \in \underline{\mathbb{R}}$ is either invertible or zero divisor.

Any differentiable function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}, f = f(a)$ is completely defined on $\mathbb{I} \subset \mathbb{R}$ such that:

$$f: \underline{\mathbb{I}} \subset \underline{\mathbb{R}} \to \underline{\mathbb{R}}, f(\underline{a}) = f(a) + \varepsilon a_0 f'(a) \tag{2}$$

Consequently, this property allows to calculate:
$\cos\,\underline{a} = \cos\,a - \varepsilon a_0 \sin\,a$; $\sin\,\underline{a} = \sin\,a + \varepsilon a_0 \cos\,a$; $\sqrt[n]{\underline{a}} = \sqrt[n]{a} + \varepsilon\frac{a_0}{n\sqrt[n]{a^{n-1}}}$; $\tan\,\underline{a} = \tan\,a + \varepsilon\frac{a_0}{\cos^2\,a}$; $\arctan\,(\underline{a}) = \arctan\,(a) + \varepsilon\frac{a_0}{1+a^2}$
.

Remark 1. *For any* $\underline{a}, \underline{b} \in \underline{\mathbb{R}}, \underline{a} = a + \varepsilon a_0 \cdot \underline{b} = b + \varepsilon b_0$ *with* $Re(\underline{a}^2 + \underline{b}^2) \neq 0$, *the atan2 function gives the dual number [5]:*

$$atan2(\underline{b}, \underline{a}) = atan2(b, a) + \varepsilon\frac{b_0 a - b a_0}{a^2 + b^2} \tag{3}$$

Proof. Let $\mathrm{atan2}(\underline{b}, \underline{a}) = \underline{\alpha}$ and $\sin\,\underline{\alpha} = \frac{\underline{b}}{\sqrt{\underline{a}^2+\underline{b}^2}}\cos\,\underline{\alpha} = \frac{\underline{a}}{\sqrt{\underline{a}^2+\underline{b}^2}}$Given that$\underline{a} = a + \varepsilon a_0, \underline{b} = b + \varepsilon b_0$ we have $\underline{a}^2 + \underline{b}^2 = a^2 + b^2 + \varepsilon(2aa_0 + 2bb_0)$, applying eq. (2) yields:

$$\begin{aligned} \frac{\underline{b}}{\sqrt{\underline{a}^2 + \underline{b}^2}} &= \frac{b}{\sqrt{a^2 + b^2}} + \varepsilon\frac{1}{\sqrt{a^2 + b^2}}\frac{b_0 a^2 - b a a_0}{a^2 + b^2} \\ \frac{\underline{a}}{\sqrt{\underline{a}^2 + \underline{b}^2}} &= \frac{a}{\sqrt{a^2 + b^2}} + \varepsilon\frac{1}{\sqrt{a^2 + b^2}}\frac{a_0 b^2 - a b b_0}{a^2 + b^2} \end{aligned} \tag{4}$$

The structure of the dual angle
$\underline{\alpha} = \alpha + \varepsilon d$impliessin $\underline{\alpha} = \sin\,\alpha + \varepsilon d \cos\,\alpha$; $\cos\,\underline{\alpha} = \cos\,\alpha - \varepsilon d \cos\,\alpha$. Through straight algebraic, it is obtained

$$\alpha = \operatorname{atan} 2(b, a)$$
$$d = \frac{b_0 a - b a_0}{a^2 + b^2}, a \neq 0 \text{ and } b \neq 0 \tag{5}$$

that confirms the previous Remark.
A direct consequenceof this Remarkis that multiplication of $\underline{a}$ and $\underline{b}$ with a dual number $\underline{k}(\operatorname{Re}(\underline{k}) \neq 0)$results in$\operatorname{atan2}(\underline{kb}, \underline{ka}) = \operatorname{atan2}(\underline{b}, \underline{a})$.

Dual vectors

Let $V_3$the 3-dimensionallinear space of free vectors in Euclidean space. The set of dual vectors is defined as

$$\underline{V_3} = V_3 + \varepsilon V_3 = \{\underline{\mathbf{a}} = \mathbf{a} + \varepsilon \mathbf{a}_0; \mathbf{a}, \mathbf{a}_0 \in V_3, \varepsilon^2 = 0, \varepsilon \neq 0\} \tag{6}$$

where $\mathbf{a} = \operatorname{Re}(\underline{\mathbf{a}})$ is the real part of $\mathbf{a}$ and $\mathbf{a}_0 = Du(\underline{a})$ the dual part. On V_3, three products are considered: scalar product ($\underline{\mathbf{a}} \cdot \underline{\mathbf{b}}$), cross product ($\underline{\mathbf{a}} \times \underline{\mathbf{b}}$) and triple scalar product ($\langle \underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}} \rangle = \underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$. Algebraically, $(\underline{V_3}, +, \underline{\cdot})$forms a free$\underline{\mathbb{R}}$-module [3].

Products for dual vectors:

- Cross product: $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = (\mathbf{a} \times \mathbf{b}) + \epsilon(\mathbf{a} \times \mathbf{b}_0 + \mathbf{a}_0 \times \mathbf{b})$.
- Dot product: $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} = (\mathbf{a} \cdot \mathbf{b}) + \epsilon(\mathbf{a} \cdot \mathbf{b}_0 + \mathbf{a}_0 \cdot \mathbf{b})$.
- Triple scalar product:

$$\langle \underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}} \rangle = \underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \epsilon[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}_0 + \mathbf{b}_0 \times \mathbf{c}) + \mathbf{a}_0 \cdot (\mathbf{b} \times \mathbf{c})]$$

The magnitude$|\underline{\mathbf{a}}|$ of any dual vector $\underline{\mathbf{a}} \in \underline{V_3}$, is the dual number satisfying $|\underline{\mathbf{a}}| \cdot |\underline{\mathbf{a}}| = \underline{\mathbf{a}} \cdot \underline{\mathbf{a}}$ and can be obtained by:

$$|\underline{\mathbf{a}}| = \begin{cases} \|\mathbf{a}\| + \varepsilon \dfrac{\mathbf{a}_0 \cdot \mathbf{a}}{\|\mathbf{a}\|}, \operatorname{Re}(\underline{\mathbf{a}}) \neq \mathbf{0} \\ \varepsilon \|\mathbf{a}_0\|, \operatorname{Re}(\underline{\mathbf{a}}) = \mathbf{0} \end{cases} \tag{7}$$

where $\|\cdot\|$ is the Euclidean norm. When the magnitude of a dual vector is 1 then the dual vector $\underline{\mathbf{a}}$ is called unit dual vector.
Theorem 1. *For any $\underline{\mathbf{a}} \in \underline{V_3}$, there exist a dual number $\underline{\alpha} \in \underline{\mathbb{R}}$and a unit dual vector $\underline{\mathbf{u}}_a \in \underline{V_3}$such that [5]*

$$\underline{\mathbf{a}} = \underline{\alpha} \underline{\mathbf{u}}_a \tag{8}$$

These components $\underline{\alpha}$ and $\underline{\mathbf{u}}_a$,are computed using$\pm\underline{\alpha} = |\underline{\mathbf{a}}|$

$$\pm\underline{\mathbf{u}}_a = \begin{cases} \dfrac{\mathbf{a}}{\|\mathbf{a}\|} + \varepsilon \dfrac{\mathbf{a} \times (\mathbf{a}_0 \times \mathbf{a})}{\|\mathbf{a}\|^3} \quad \operatorname{Re}(\underline{a}) \neq \mathbf{0} \\ \dfrac{\mathbf{a}_0}{\|\mathbf{a}_0\|} + \varepsilon \mathbf{v} \times \dfrac{\mathbf{a}_0}{\|\mathbf{a}_0\|}, \forall \mathbf{v} \in V_3 \ \operatorname{Re}(\underline{\mathbf{a}}) = \mathbf{0} \end{cases} \tag{9}$$

Also, for $Re(\underline{\mathbf{a}}) \neq \mathbf{0}$, $\underline{\alpha}$, and $\underline{\mathbf{u}}_a$ are unique up to a sign change.
This theorem, proven in[3] establishes a correspondencebetween any dual vector $\underline{\mathbf{a}} \in \underline{V_3}$, with

$\mathrm{Re}(\underline{\mathbf{a}}) \neq 0$and a *labeled* directed line in the Euclidean three dimensional space (Figure l) defined by the parametric equation: $\mathbf{r} = \frac{\mathbf{a}\times\mathbf{a}_0}{\|\mathbf{a}\|^2} + \lambda\frac{\mathbf{a}}{\|\mathbf{a}\|}, \forall\lambda \in \mathbb{R}$. When$\mathrm{Re}(\underline{\mathbf{a}}) = \mathbf{0}$ the parametric equation is $\mathbf{r} = \mathbf{v} + \lambda\frac{\mathbf{a}_0}{\|\mathbf{a}_0\|}, \forall\mathbf{v} \in V_3, \forall\lambda \in \mathbb{R}$.

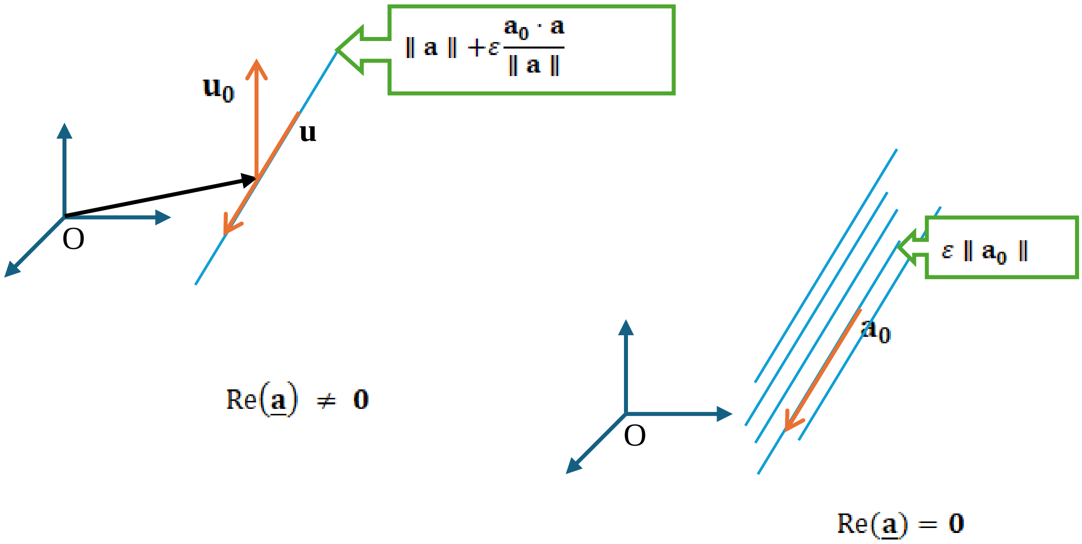

Figure 1: Illustration of how dual vectors are represented geometrically using directed lines with associated labels

Remark 2. The relationship between any *pair of dual vectors* $\underline{\mathbf{b}}, \underline{\mathbf{a}} \in \underline{V}_3 \setminus \{\underline{\mathbf{0}}\}$ can be described by the dual number $\underline{\alpha} = \alpha + \varepsilon d$ *(Figure 2), denoted the dual angle, which is calculated using:*

$$\underline{\alpha} = \mathrm{atan2}\left(\left|\underline{\mathbf{u}}_b \times \underline{\mathbf{u}}_a\right|, \underline{\mathbf{u}}_a \cdot \underline{\mathbf{u}}_b\right) \tag{10}$$

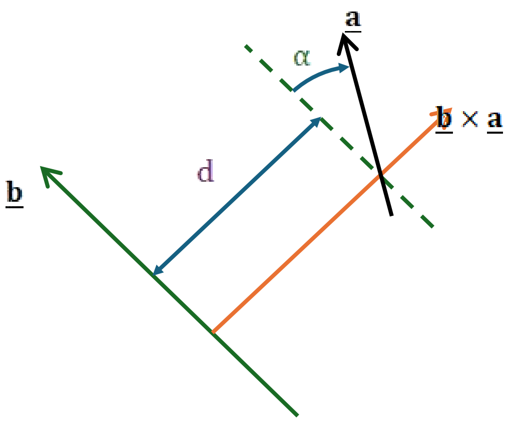

Figure 2: Geometrical representation of dual angles. $\underline{\alpha} = \alpha + \varepsilon d$

where $\underline{\mathbf{u}}_b$ *and* $\underline{\mathbf{u}}_a$ *are the unit dual vectors of* $\underline{\mathbf{b}}$ *and* $\underline{\mathbf{a}}$ *recovered as in Theorem 1. If* $Re(\underline{\mathbf{a}}) \neq 0$ *and* $Re(\underline{\mathbf{b}}) \neq 0$ *then the dual angle can be directly computed using:*

$$\underline{\alpha} = \mathrm{atan2}(|\underline{\mathbf{a}} \times \underline{\mathbf{b}}|, \underline{\mathbf{a}} \cdot \underline{\mathbf{b}}) \tag{11}$$

3. Classical Volume Computation using Determinants

Volume calculations in Euclidean three-dimensional space can be performed using linear algebraic tools such as determinants and matrices [10].

In a three-dimensional Euclidean space $\mathbb{R}^3$, vectors are represented as ordered triples:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 \quad (12)$$

The dot product and cross product are fundamental operations that help define volume computations.

Given three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$, the volume of the parallelepiped they define is given by the absolute value of the determinant of the matrix formed by these vectors as columns:

$$V = \left| \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \right| \quad (13)$$

For a general 3×3 matrix:

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad (14)$$

The volume can also be related to the cross product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\det A| \quad (15)$$

Where A is the 3×3 matrix with columns $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The determinant encodes orientation and scaling.

Here, $\mathbf{b} \times \mathbf{c}$ results in a vector perpendicular to the plane spanned by $\mathbf{b}$ and $\mathbf{c}$, and the dot product with $\mathbf{a}$ gives the scalar triple product, equivalent to the determinant.

For a general shape in 3D space, volume calculations often involve integration and can be expressed using determinants and matrices. The volume integral can be formulated as:

$$V = \iiint_D dV \quad (16)$$

where D represents the domain enclosed by the shape. If the shape is defined parametrically using a transformation matrix J (the Jacobian matrix), the volume element is given by:

$$dV = |\det(J)| du dv dw \quad (17)$$

For a transformation $\mathbf{x} = T(u, v, w)$, the Jacobian matrix J is:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{bmatrix} \quad (18)$$

The total volume is then obtained by integrating over the parameter space:

$$V = \iiint_D |\det(J)| dudvdw \quad (19)$$

Let a 3D space $\mathcal{V}$ be described by a position vector $\mathbf{r}(u,v,w)$, where u,v,w are parameters in a domain $D \subseteq \mathbb{R}^3$:

$$\mathbf{r}(u,v,w) = x(u,v,w)\mathbf{i} + y(u,v,w)\mathbf{j} + z(u,v,w)\mathbf{k} \quad (20)$$

The volume is "sliced" into infinitesimal parallelepipeds spanned by the partial derivatives of vector $\mathbf{r}$:

$$\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial w} \quad (21)$$

The volume of an infinitesimal region is given by the scalar triple product of the tangent vectors:

$$dV = \left| \frac{\partial \mathbf{r}}{\partial u} \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w} \right) \right| dudvdw \quad (22)$$

This represents the volume of a tiny parallelepiped at each point (u,v,w).

The total volume of $\mathcal{V}$ is computed by integrating dV over D:

$$\text{Volume}(\mathcal{V}) = \iiint_D \left| \frac{\partial \mathbf{r}}{\partial u} \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w} \right) \right| dudvdw \quad (23)$$

This generalizes to any parametrization, including Cartesian, cylindrical, or spherical coordinates.

Coordinate Transformations

For a coordinate change $(x,y,z) \rightarrow (u,v,w)$, the volume element transforms via the Jacobian determinant J

$$dV = |J| dudvdw \quad (24)$$

where

$$J = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \tag{25}$$

The Jacobian J is equivalent to the scalar triple product $\frac{\partial \mathbf{r}}{\partial u} \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)$.

Particular cases

- Cartesian Coordinates: $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$:

$dV = dxdydz$

- Cylindrical Coordinates: $\mathbf{r}(r, \theta, z) = r\cos\,\theta\mathbf{i} + r\sin\,\theta\mathbf{j} + z\mathbf{k}$:

$dV = rdrd\theta dz$

- Spherical Coordinates: $\mathbf{r}(\rho, \phi, \theta) = \rho\sin\,\phi\cos\,\theta\mathbf{i} + \rho\sin\,\phi\sin\,\theta\mathbf{j} + \rho\cos\,\phi\mathbf{k}$:

$dV = \rho^2 \sin\,\phi d\rho d\phi d\theta$

4. Volume Computation Using Dual Vectors

Given three dual vectors $\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}} \in \underline{V} = \underline{V}_3$, we define the volume using the determinant

$$\underline{V} = \det[\underline{\mathbf{a}} \quad \underline{\mathbf{b}} \quad \underline{\mathbf{c}}] \tag{25}$$

Since each dual vector is represented as:

$$\underline{\mathbf{a}} = \mathbf{a} + \varepsilon\mathbf{a}_0, \underline{\mathbf{b}} = \mathbf{b} + \varepsilon\mathbf{b}_0, \underline{\mathbf{c}} = \mathbf{c} + \varepsilon\mathbf{c}_0 \tag{26}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V_3$ are the real parts, and $\mathbf{a}_0, \mathbf{b}_0, \mathbf{c}_0$ are the dual parts.
Expanding the determinant:

$$\mathrm{V}_{\mathrm{dual}} = \det[\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}] = \det[\mathbf{a} + \varepsilon\mathbf{a}_0 \quad \mathbf{b} + \varepsilon\mathbf{b}_0 \quad \mathbf{c} + \varepsilon\mathbf{c}_0] \tag{27}$$

Using the determinant linearity and properties of dual numbers ($\varepsilon^2 = 0$), we obtain:

$$\mathrm{V}_{\mathrm{dual}} = \det[\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}] + \varepsilon\det\begin{bmatrix} \mathbf{a}_0 & \mathbf{b} & \mathbf{c} \\ \mathbf{a} & \mathbf{b}_0 & \mathbf{c} \\ \mathbf{a} & \mathbf{b} & \mathbf{c}_0 \end{bmatrix} \tag{28}$$

This result gives the dual volume of the parallelepiped, where:

- The real part is the standard volume of the parallelepiped spanned by $\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}$.
- The dual part encodes perturbations - these can represent small shifts due to land movement, construction deviations, or measurement errors. If the dual part shows changes over time, it indicates land movement

Volume of a General Shape in 3D Space using Dual Vectors

For a general shape, we use an integral approach. The volume is computed as:

$$V_{\mathrm{dual}} = \iiint_D |\det(J_{\mathrm{dual}})| dudvdw \tag{29}$$

where J_{dual} is the dual Jacobian matrix.

Expanding the determinant, we obtain:

$$\det(J_{\mathrm{dual}}) = \det(J) + \varepsilon \sum_{i=1}^{3} \det(J_i) \tag{30}$$

where J is the real Jacobian matrix, and each J_i is obtained by replacing the i column of J with the corresponding dual part column.

Parametrization with Dual Vectors

Let the shape be parametrized by a dual vector-valued function:

$$\mathbf{r}(u,v,w) = \mathbf{r}_{\mathrm{Re}}(u,v,w) + \epsilon \mathbf{r}_{\mathrm{Du}}(u,v,w) \tag{31}$$

where $\mathbf{r}_{\mathrm{Re}}, \mathbf{r}_{\mathrm{Du}} \in V_3$ are real and dual components.

Infinitesimal Volume Element

The differential volume element is derived from the dual triple product of the tangent vectors:

$$dV = \left|\left\langle \frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial w} \right\rangle\right| dudvdw \tag{32}$$

Expanding this:

$$dV = |\mathbf{J}_{\mathrm{Re}} + \epsilon \mathbf{J}_{\mathrm{Du}}| dudvdw \tag{33}$$

where $\mathbf{J}_{\mathrm{Re}} = \mathbf{r}_u \cdot (\mathbf{r}_v \times \mathbf{r}_w)$ is the classical Jacobian determinant, and $\mathbf{J}_{\mathrm{Du}}$ captures dual corrections from $\mathbf{r}_{\mathrm{Du}}$.

Thus, the total dual volume is:

$$V_{dad} = \iiint_D \left(|\det(J)| + \varepsilon \sum_{i=1}^{3} |\det(J_i)| \right) dudvdw \tag{34}$$

Total Volume

Integrate the dual volume element over the parameter domain $D \subseteq \mathbb{R}^3$, we obtain:

$$\mathrm{Volume}(\mathcal{V}) = \iiint_D |\mathbf{J}_{\mathrm{Re}} + \epsilon \mathbf{J}_{\mathrm{Du}}| dudvdw \tag{35}$$

This separates into:

$$\text{Volume}(\mathcal{V}) = \iiint_D |\mathbf{J}_{\text{Re}} + \epsilon \mathbf{J}_{\text{Du}}| du dv dw \tag{36}$$

5. Comparative Analysis

Table 1. Comparative analysisin volume calculating using determinants and dual vectors

Method	Advantages	Limitations
Determinants	Computationally straightforward in a basis	Necessity of coordinates
Dual Vectors	Intrinsically geometric, coordinate-free, applicable in differential geometry	More abstract, requires understanding of advanced algebra

The dual volume calculation is especially useful in cadastre (land surveying), where volumetric computations are crucial for:parcel volume calculations (underground structures, excavation sites).building volume measurements (for property valuation and taxation).deformation analysis (monitoring subsidence, construction deviations, and terrain shifts). The dual part of the volume helps model:small deformations in land measurements, comparisons of pre- and post-construction volumes geometric variations in GIS and CAD modeling.

By integrating dual numbers into cadastral computations, we obtain more refined volumetric analyses that include both standard volume and small perturbations due to land shifts or measurement uncertainties.

6. Conclusions

This paper highlights that while determinants offer an efficient approach for volume computation [11] in a fixed coordinate system, dual algebra provides a coordinate-free alternative. These results have practical implications in 3D cadastre, geospatial analysis, and land surveying, where coordinate-independent volume calculations are essential for terrain modeling, boundary delineation, and volumetric land assessments. The adaptability of dual algebra in these fields suggests its potential for further research[7] and integration into computational tools for surveying and mapping sciences.

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